An introduction to diagram categories

Jonathan Comes

March 23, 2017

Contents

Part I

Discrete diagram categories

Chapter 1

Diagrams

1.1 Partition diagrams

The first diagrams we are going to consider are partition diagrams. The reason to start with partition diagrams is that many interesting families of diagrams can be viewed as special types of partition diagrams.

We define a *partition* of a set A to be a collection of mutually disjoint nonempty subsets of A whose union is A. The disjoint subsets are called *parts* of the partition. For example, $\{1,4\},\{2\},\{3,5\}$ is a partition of $\{1,2,3,4,5\}$ into three parts. In particular (or by convention) there is a unique partition of the empty set, namely the one with zero parts. There is also a unique partition of any singleton set $\{x\}$; it has one part. A set with two elements $\{a,b\}$ admits exactly two partitions: $\{a\},\{b\}$ and $\{a,b\}$.

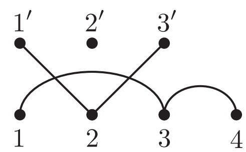
Exercise 1.1. (Bell numbers) The number of partitions of an n-element set is called the *nth Bell number*. List the Bell numbers for $n = 0,1,2,3,4$. You can check your answers by looking at Bell's triangle (similar to Pascal's triangle). Here's the top of Bell's triangle:

$$\begin{array}{ccccc} 1 & & & & \\ 1 & 2 & & & \\ 2 & 3 & 5 & & \\ 5 & ? & ? & ? & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

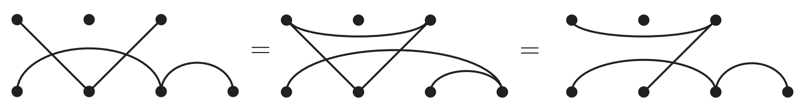
Every entry in the leftmost column is the same as the rightmost entry in the previous row. All other entries are obtained by summing the entry 1 step to the left with the entry 1 step to the left and 1 step up. Construct a couple more rows. The leftmost column in Bell's triangle lists the Bell numbers. Notice that they grow very fast.

Given $n \in \mathbb{Z}_{\geq 0}$, write $[n] = \{i \in \mathbb{N} \ : \ 1 \leq i \leq n\}$ and $[n]' = \{i' \ : \ i \in [n]\}$. For example, $[3] = \{1,2,3\}$ and $[4]' = \{1',2',3',4'\}$. Moreover, $[0] = \varnothing = [0]'$.

Now, given $m, n \in \mathbb{Z}_{\geq 0}$, a *partition diagram of type $m \to n$* is a diagrammatic representation of a partition of $[m] \cup [n]'$ obtained by (1) listing m vertices in a bottom row and n vertices in a top row; (2) labelling the bottom row with the elements of $[m]$ and the top row with elements in $[n]'$, both in increasing order from left to right; and (3) connecting vertices with edges in such a way that there is a path between two vertices exactly when those vertices are in the same part of the partition. We require that all edges be drawn in between the two rows of vertices. For example, the following diagram is of type $4 \to 3$ and corresponds to the partition $\{1, 3, 4\}, \{1', 2, 3'\}, \{2'\}$:



Since the vertices' labels are always increasing from left to right, we can draw our diagrams without labelling the vertices without loosing any information. We declare two partition diagrams to be equal if they represent the same partition. For instance:

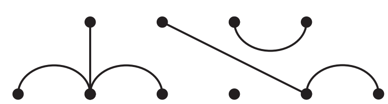


Exercise 1.2. Draw the partition diagram of type $4 \to 4$ corresponding to the partition $\{1\}, \{2, 2', 3'\}, \{3, 4'\}, \{4, 1'\}$.

Exercise 1.3. Draw all the partition diagrams of types $2 \to 1$ and $0 \to 3$.

1.2 Special types of partition diagrams

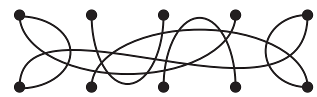
A partition diagram is called *non-crossing** if it can be drawn in such a way that no two edges cross (aside from at vertices). Here's an example of a non-crossing partition diagram:



Note that partition diagrams with crossings can be non-crossing since there are multiple ways to draw the same diagram. As long as there is one way to draw the diagram without crossings, the diagram is non-crossing.

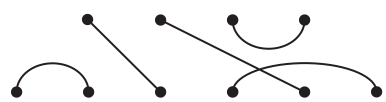
*Non-crossing partition diagrams are also called *planar*.

Exercise 1.4. Draw a better picture of the following partition diagram to show that it is non-crossing:



Exercise 1.5. How many non-crossing partition diagrams are there of type $4 \to 0$?
Hint: It might be easier to count the number of non-non-crossing partition diagrams and use Exercise 1.1.

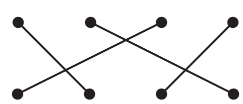
A partition diagram is called a *Brauer diagram* if all of the parts in the corresponding partition have exactly 2 elements. In other words, a Brauer diagram is a partition diagram where each vertex is connected to exactly one other vertex. Here's an example:



Exercise 1.6. Draw all the Brauer diagrams of type $1 \to 3$. How many Brauer diagrams are there of type $2 \to 3$?

Exercise 1.7. How many Brauer diagrams are there of type $m \to n$?

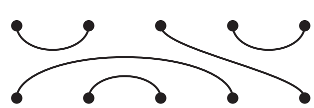
An edge in a Brauer diagram is called a cap if its endpoints are both bottom vertices, a cup if they're both top vertices, and a propagating edge otherwise. A Brauer diagram with only propagating edges (i.e. no caps or cups) is called a *permutation diagram*. For example, here's a permutation diagram:



Exercise 1.8. Draw all the permutation diagrams of type $3 \to 3$.

Exercise 1.9. How many permutation diagrams are there of type $m \to n$?

A *Temperley-Lieb diagram* is a non-crossing Brauer diagram. Here's one:



Exercise 1.10. Draw all the Temperley-Lieb diagrams of type $4 \to 2$.

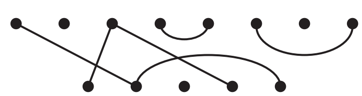
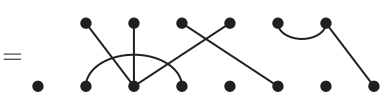
Exercise 1.11. (Catalan numbers) The *nth Catalan number*, denoted C_n, can be defined recursively by letting $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ for $n \geq 0$. Use that formula to find the first few Catalan numbers. Prove that C_n is the number of Temperley-Lieb diagrams of type $2n \to 0$.

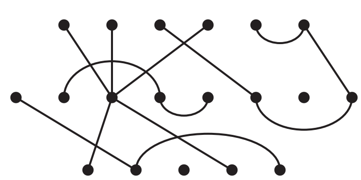
Exercise 1.12. Using the following pictures as a guide, describe a bijection between the set of all non-crossing partition diagrams of type $m \to n$ and the set of all Temperley-Lieb diagrams of type $2m \to 2n$.

1.3 Stacking products

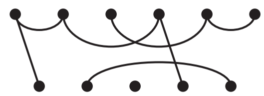
We can "multiply" two diagrams together by stacking them. There are two flavors of multiplication corresponding to the two ways to stack: vertically and horizontally. We will start with vertical stacking. From now on we will write $D : m \to n$ to mean that D is a partition diagram of type $m \to n$.

Given partition diagrams $D_1 : l \to m$ and $D_2 : m \to n$, we write $D_2 \star D_1$ for the diagram obtained by stacking D_2 on top of D_1 (identifying the m middle vertices). For example,

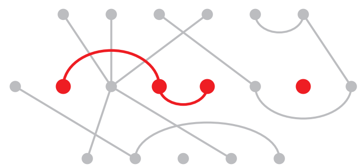
if $D_1 =$  and $D_2 =$ 

then $D_2 \star D_1 =$  .

Now, we let $D_2 \bullet D_1$ denote the partition diagram of type $l \to n$ such that there is a path between two vertices in $D_2 \bullet D_1$ if and only if there is a path between the corresponding vertices in $D_2 \star D_1$. For instance, in the example above

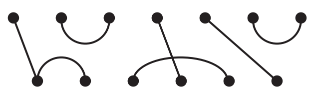
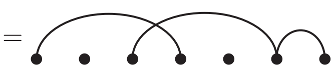
$D_2 \bullet D_1 =$  .

In this example, there are two connected components in $D_2 \star D_1$ that did not contribute to $D_2 \bullet D_1$, highlighted below:



It will be important for us to keep track of the number of such components. In general, we let $\ell(D_2, D_1)$ denote the number of such components. In other words, $\ell(D_2, D_1)$ is the number of connected components in $D_2 \star D_1$ minus the number of connected components in $D_2 \bullet D_1$.

Exercise 1.13. Compute $D_2 \bullet D_1$ and $\ell(D_2, D_1)$ where

$D_1 =$  $D_2 =$ 

Exercise 1.14. Suppose $D_1 : l \to m$ and $D_2 : m \to n$. Prove the following:

1. If D_1 and D_2 are non-crossing partition diagrams, then $D_2 \bullet D_1$ is too.
2. If D_1 and D_2 are Brauer diagrams, then $D_2 \bullet D_1$ is too.
3. If D_1 and D_2 are permutation diagrams, then $D_2 \bullet D_1$ is too.
4. If D_1 and D_2 are Temperley-Lieb diagrams, then $D_2 \bullet D_1$ is too.

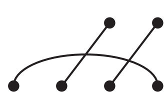
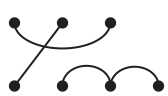
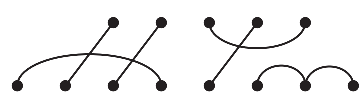
Exercise 1.15. Suppose $D_1 : l \to m$ and $D_2 : m \to n$. Prove that if either D_1 or D_2 is a permutation diagram, then $\ell(D_2, D_1) = 0$. Is the converse true?

Exercise 1.16. (Identity diagrams) Find a diagram $1_n : n \to n$ for each $n \in \mathbb{Z}_{\geq 0}$ such that $1_n \bullet D = D = D \bullet 1_m$ whenever $D : m \to n$.

Exercise 1.17. Suppose $D_1 : k \to l, D_2 : l \to m$, and $D_3 : m \to n$. Prove:

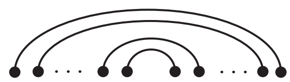
1. $(D_3 \bullet D_2) \bullet D_1 = D_3 \bullet (D_2 \bullet D_1)$.
2. $\ell(D_3, D_2) + \ell(D_3 \bullet D_2, D_1) = \ell(D_3, D_2 \bullet D_1) + \ell(D_2, D_1)$.

The *tensor product* of two partition diagrams $D_1 : m_1 \to n_1$ and $D_2 : m_2 \to n_2$, denoted $D_1 \otimes D_2$, is the partition diagram of type $m_1 + m_2 \to n_1 + n_2$ obtained by stacking D_1 to the left of D_2. For example,

 $\otimes$  $=$  .

Exercise 1.18. Repeat Exercise 1.14 replacing $\bullet$'s with $\otimes$'s.

Exercise 1.19. Let $\Cap_n : 2n \to 0$ and $\Cup_n : 0 \to 2n$ be the following diagrams:

$\Cap_n =$  $\Cup_n =$ 

Prove that the map $\{D : m \to n\} \to \{D : m + n \to 0\}$ given by

$$D \mapsto \Cap_n \bullet (D \otimes 1_n)$$

is a bijection with inverse

$$D \mapsto (D \otimes 1_n) \bullet (1_m \otimes \Cup_n).$$

Explain why the bijection is still valid if we require the D's to be any one of the special types: non-crossing, Brauer, Temperley-Lieb. Use this bijection along with Exercises 1.11 and 1.12 to give formulas for the number of Temperley-Lieb diagrams of type $m \to n$ and non-crossing partition diagrams of type $m \to n$ in terms of Catalan numbers.

Chapter 2

Diagram categories

2.1 Definition of a category

We are about to define the term *category*, and the definition can be a lot to swallow. As motivation, consider the following three operations:

1. Vertical stacking of partition diagrams (see §1.3). Given $D_1 : k \to l$ and $D_2 : m \to n$, the product $D_2 \bullet D_1$ makes sense if and only if $l = m$.

2. Matrix multiplication. Given an $l \times k$ matrix A_1 and an $n \times m$ matrix A_2, the matrix product $A_2 A_1$ makes sense if and only if $l = m$.

3. Function composition. Given functions $f_1 : K \to L$ and $f_2 : M \to N$, the composition $f_2 \circ f_1$ makes sense if and only if the sets $L = M$.

There are many applications of category theory, but the main reason that we work with categories is they provide a uniform setting for studying operations on stuff where the operation is only defined when the stuff is compatible. In the language of category theory the stuff (diagrams, matrices, functions) are called *morphisms*; the operation (vertical stacking, matrix multiplication, function composition) is called *composition*; and the type of a morphism ($m \to n$, $n \times m$, $M \to N$) is described by a pair of *objects* (nonnegative integers, natural numbers, sets). Keep the three examples above in mind when reading the following:

Definition 2.1. A *category* $\mathcal{C}$ consist of the following data:

- A class of *objects* $\mathrm{Ob}\,\mathcal{C}$.

- A class of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(m, n)$ for every pair of objects (m, n). Given $f \in \mathrm{Hom}_{\mathcal{C}}(m, n)$ we will often write $f : m \to n$. We call m the *domain* of f and n the *target* of f.

- A *composition map* $\mathrm{Hom}_{\mathcal{C}}(m, n) \times \mathrm{Hom}_{\mathcal{C}}(l, m) \to \mathrm{Hom}_{\mathcal{C}}(l, n)$ for every triple of objects (l, m, n). We will use the notation $f \circ g : l \to n$ for the composition of $f : m \to n$ and $g : l \to m$.

The data above must satisfy the following axioms:

(C1) Every morphism has a unique domain and target.

(C2) (Identity morphisms) For all $n \in \operatorname{Ob}\mathcal{C}$ there exists $\mathrm{id}_n \in \operatorname{Hom}_{\mathcal{C}}(n,n)$ such that $f \circ \mathrm{id}_n = f$ and $\mathrm{id}_n \circ g = g$ for all $f : n \to m$ and $g : m \to n$.

(C3) (Composition is associative) $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms $f : m \to n, g : l \to m, h : k \to l$.

Here are the formal definitions of the examples of categories discussed above:

Example 2.2. (Partition diagrams) Let $\mathcal{P}$ denote the category with

- $\operatorname{Ob}\mathcal{P} = \mathbb{Z}_{\geq 0}$.
- $\operatorname{Hom}_{\mathcal{P}}(m,n) = \{\text{partition diagrams of type } m \to n\}$ for each $m, n \in \mathbb{Z}_{\geq 0}$.
- The composition map is given by vertical stacking:

$$\begin{array}{rcl} \operatorname{Hom}_{\mathcal{P}}(m,n) \times \operatorname{Hom}_{\mathcal{P}}(l,m) & \to & \operatorname{Hom}_{\mathcal{P}}(l,n) \\ (D_2, D_1) & \mapsto & D_2 \bullet D_1 \end{array}$$

Axiom (C1) follows from the fact that each partition diagram has a unique number of bottom vertices (the domain) and a unique number of top vertices (the target). Axioms (C2) and (C3) follow from Exercises 1.16 and 1.17.1 respectively.

Example 2.3. (Matrices over $\mathbb{C}$) Let **Mat** denote the category with

- $\operatorname{Ob}\mathbf{Mat} = \mathbb{Z}_{\geq 0}$.
- $\operatorname{Hom}_{\mathbf{Mat}}(m,n) = \{n \times m \text{ matrices with entries in } \mathbb{C}\}$ for all $m, n \in \mathbb{Z}_{\geq 0}$.
- Composition is given by matrix multiplication:

$$\begin{array}{rcl} \operatorname{Hom}_{\mathbf{Mat}}(m,n) \times \operatorname{Hom}_{\mathbf{Mat}}(l,m) & \to & \operatorname{Hom}_{\mathbf{Mat}}(l,n) \\ (A_2, A_1) & \mapsto & A_2 A_1 \end{array}$$

Remark 2.4. By convention, we declare that there is a unique $0 \times n$ matrix and a unique $n \times 0$ matrix for all $n \in \mathbb{Z}_{\geq 0}$, which we denote by 0 in all cases. Multiplying any matrix by these new zero matrices always results in 0.

Example 2.5. (The category of sets) Let **Set** denote the category with

- $\operatorname{Ob}\mathbf{Set}$ is the collection of all sets.
- $\operatorname{Hom}_{\mathbf{Set}}(M,N) = \{\text{functions from } M \text{ to } N\}$ for all sets M and N.
- Composition is the usual composition of functions:

$$\begin{array}{rcl} \operatorname{Hom}_{\mathbf{Set}}(M,N) \times \operatorname{Hom}_{\mathbf{Set}}(L,M) & \to & \operatorname{Hom}_{\mathbf{Set}}(L,N) \\ (f_2, f_1) & \mapsto & f_2 \circ f_1 \end{array}$$

Exercise 2.6. Verify that **Mat** and **Set** satisfy axioms (C1)-(C3).

Exercise 2.7. (The category of relations) Given two sets M and N, a *relation* R from M to N is a subset $R \subseteq M \times N$. We write $R : M \to N$ to indicate that R is a relation from M to N. We define the composition of two relations $R_1 \subseteq L \times M$, $R_2 \subseteq M \times N$ to be the subset $R_2 \circ R_1 \subseteq L \times N$ given by $R_2 \circ R_1 = \{(x, z) \in L \times N \ : \ \exists y \in M \text{ with } (x, y) \in R_1 \text{ and } (y, z) \in R_2\}$. Let **Rel** denote the category with

- $\mathrm{Ob}\,\mathbf{Rel}$ is the collection of all sets.
- $\mathrm{Hom}_{\mathbf{Rel}}(M, N) = \{\text{relations from } M \text{ to } N\}$ for all sets M and N.
- Composition is the composition of relations defined above:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{Rel}}(M, N) \times \mathrm{Hom}_{\mathbf{Rel}}(L, M) & \to & \mathrm{Hom}_{\mathbf{Rel}}(L, N) \\ (R_2, R_1) & \mapsto & R_2 \circ R_1 \end{array}$$

Note that we declare two relations $R : M \to N$ and $R' : M' \to N'$ to be equal if and only if $M = M'$, $N = N'$, and $R = R'$ (equal as sets). It follows that **Rel** satisfies axiom (C1). Verify that **Rel** satisfies (C2) and (C3).

2.2 Subcategories of $\mathcal{P}$

Of the categories introduced in the previous section, the diagram category $\mathcal{P}$ is the main focus of these notes. If we restrict ourselves to the various special types of partition diagram discusses in §1.2, we obtain other interesting diagram categories. These diagram categories are best described as "subcategories" of $\mathcal{P}$. Let's be precise about what that term means:

Definition 2.8. Suppose $\mathcal{C}$ is a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ (written $\mathcal{D} \subseteq \mathcal{C}$) consists of the following data:

- A class of objects $\mathrm{Ob}\,\mathcal{D} \subseteq \mathrm{Ob}\,\mathcal{C}$.
- A class of morphisms $\mathrm{Hom}_{\mathcal{D}}(m, n) \subseteq \mathrm{Hom}_{\mathcal{C}}(m, n)$ for each pair of objects $m, n \in \mathrm{Ob}\,D$.

The data above must satisfy the following:

(SC1) ($\mathcal{D}$ has identities) $\mathrm{id}_n \in \mathrm{Hom}_{\mathcal{D}}(n, n)$ for every $n \in \mathrm{Ob}\,\mathcal{D}$.

(SC2) ($\mathcal{D}$ is closed under composition) The composition $f \circ g \in \mathrm{Hom}_{\mathcal{D}}(l, n)$ whenever $f \in \mathrm{Hom}_{\mathcal{D}}(m, n)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(l, m)$.

Exercise 2.9. Show that subcategories are indeed categories. More precisely, assume $\mathcal{D}$ is a subcategory of $\mathcal{C}$. Show the composition maps for $\mathcal{C}$ restrict to composition maps for $\mathcal{D}$, and $\mathcal{D}$ satisfies (C1)-(C3).

The diagram categories $\mathcal{NC}, \mathcal{B}, \mathcal{TL}$, and $\mathcal{S}$

The categories $\mathcal{NC}, \mathcal{B}, \mathcal{TL}$, and $\mathcal{S}$ are defined to be the subcategories of $\mathcal{P}$ with

$$\operatorname{Ob}\mathcal{NC} = \operatorname{Ob}\mathcal{B} = \operatorname{Ob}\mathcal{TL} = \operatorname{Ob}\mathcal{S} = \mathbb{Z}_{\geq 0}$$

and morphisms given by

$$\begin{aligned} \operatorname{Hom}_{\mathcal{NC}}(m,n) &= \{\text{Non-crossing partition diagrams of type } m \to n\}, \\ \operatorname{Hom}_{\mathcal{B}}(m,n) &= \{\text{Brauer diagrams of type } m \to n\}, \\ \operatorname{Hom}_{\mathcal{TL}}(m,n) &= \{\text{Temperley-Lieb diagrams of type } m \to n\}, \\ \operatorname{Hom}_{\mathcal{S}}(m,n) &= \{\text{permutation diagrams of type } m \to n\}. \end{aligned}$$

Indeed, each of $\mathcal{NC}, \mathcal{B}, \mathcal{TL}$, and $\mathcal{S}$ has identities (hence (SC1) is satisfied), and by Exercise 1.14 each is closed under composition (hence (SC2) is satisfied).

Even and odd subcategories of $\mathcal{B}$ and $\mathcal{TL}$

In Exercise 1.7 you ought to have shown that there are no Brauer diagrams (hence no Temperley-Lieb diagrams) of type $m \to n$ unless m and n are both odd or both even. In some sense, this means that the categories $\mathcal{B}$ and $\mathcal{TL}$ each split into an even piece and an odd piece. Let us be more precise:

We start with $\mathcal{B}$. The categories $\mathcal{B}^{ev}$ and $\mathcal{B}^{odd}$ are defined as the subcategories of $\mathcal{B}$ with objects

$$\begin{aligned} \operatorname{Ob}\mathcal{B}^{ev} &= \{2n \ : \ n \in \mathbb{Z}_{\geq 0}\}, \\ \operatorname{Ob}\mathcal{B}^{odd} &= \{2n+1 \ : \ n \in \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

and morphisms

$$\begin{aligned} \operatorname{Hom}_{\mathcal{B}^{ev}}(2m,2n) &= \operatorname{Hom}_{\mathcal{B}}(2m,2n), \\ \operatorname{Hom}_{\mathcal{B}^{odd}}(2m+1,2n+1) &= \operatorname{Hom}_{\mathcal{B}}(2m+1,2n+1). \end{aligned}$$

Note that both of these subcategories are obtained from $\mathcal{B}$ by selecting a subset of the objects, and picking *all* the morphisms between the selected objects. Such subcategories get a special name:

Definition 2.10. A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is called *full* if $\operatorname{Hom}_{\mathcal{D}}(m,n) = \operatorname{Hom}_{\mathcal{C}}(m,n)$ for all $m,n \in \operatorname{Ob}\mathcal{D}$.

Thus, $\mathcal{B}^{ev}$ and $\mathcal{B}^{odd}$ are both full subcategories of $\mathcal{B}$. Not all subcategories are full. For instance, $\mathcal{B}$ is not a full subcategory of $\mathcal{P}$ since there are partition diagrams which are not Brauer diagrams (i.e. there exist $m,n \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{Hom}_{\mathcal{B}}(m,n) \neq \operatorname{Hom}_{\mathcal{P}}(m,n)$). One nice property of full subcategories is that they can be defined without checking (SC1) and (SC2) thanks to the following:

Proposition 2.11. *Let $\mathcal{C}$ be an arbitrary category. Suppose $\mathcal{D}$ is a collection of objects and morphisms in $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{D}}(m,n) = \operatorname{Hom}_{\mathcal{C}}(m,n)$ for all $m,n \in \operatorname{Ob}\mathcal{D}$. Then $\mathcal{D}$ is a full subcategory of $\mathcal{C}$.*

Proof. If we can show $\mathcal{D}$ is a subcategory of $\mathcal{C}$, then it will be full by the definition of full. Hence, we must just check that $\mathcal{D}$ satisfies (SC1) and (SC2).

For (SC1), let $n \in \operatorname{Ob}\mathcal{D}$. Then $n \in \operatorname{Ob}\mathcal{C}$ since $\operatorname{Ob}\mathcal{D} \subseteq \operatorname{Ob}\mathcal{C}$. Since $\mathcal{C}$ is a category, $\mathrm{id}_n \in \operatorname{Hom}_{\mathcal{C}}(n,n) = \operatorname{Hom}_{\mathcal{D}}(n,n)$, as desired.

For (SC2), suppose $f \in \operatorname{Hom}_{\mathcal{D}}(m,n)$ and $g \in \operatorname{Hom}_{\mathcal{D}}(l,m)$ for some objects $l,m,n \in \operatorname{Ob}\mathcal{D}$. Then $f \in \operatorname{Hom}_{\mathcal{C}}(m,n)$ and $g \in \operatorname{Hom}_{\mathcal{C}}(l,m)$ by our assumption on $\mathcal{D}$. Since $\mathcal{C}$ is a category, $f \circ g \in \operatorname{Hom}_{\mathcal{C}}(l,n) = \operatorname{Hom}_{\mathcal{D}}(l,n)$, as desired. $\square$

In particular, the previous proposition verifies that $\mathcal{B}^{ev}$ and $\mathcal{B}^{odd}$ are indeed subcategories of $\mathcal{B}$. Moreover, that proposition implies that every subset of objects in a category determines a unique full subcategory. We will exploit this fact in order to define some categories in a new, more concise way. For example, we define $\mathcal{TL}^{ev}$ and $\mathcal{TL}^{odd}$ to be the full subcategories of $\mathcal{TL}$ with objects

$$\begin{aligned} \operatorname{Ob}\mathcal{TL}^{ev} &= \{2n \ : \ n \in \mathbb{Z}_{\geq 0}\}, \\ \operatorname{Ob}\mathcal{TL}^{odd} &= \{2n+1 \ : \ n \in \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

2.3 Endomorphisms, isomorphisms, and idempotents

In this section we discuss various adjectives for morphisms. The terminology developed in this section be used throughout the rest of the notes. However, the reason we are introducing these new terms now is to get some more practice working with categories.

Endomorphisms

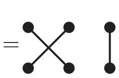
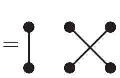
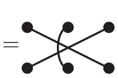
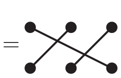
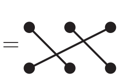
A morphism f in a category is called an *endomorphism* if it has the same domain and target, i.e. $f : n \to n$. We write $\operatorname{End}_{\mathcal{C}}(n) = \operatorname{Hom}_{\mathcal{C}}(n,n)$ for the collection of all *endomorphisms of* n. For example:

1. Endomorphisms in $\mathcal{P}$ are diagrams with the same number of top and bottom vertices. For instance, $\operatorname{End}_{\mathcal{P}}(5)$ is the set of all partition diagrams of type $5 \to 5$.

2. Endomorphisms in **Mat** are square matrices. For example $\operatorname{End}_{\mathbf{Mat}}(5)$ is the set of all 5×5 matrices.

3. Endomorphisms in **Set** are functions whose domain and target are equal.

Notice that endomorphism sets are closed under composition. In other words, if $f, g \in \operatorname{End}_{\mathcal{C}}(n)$ then $f \circ g \in \operatorname{End}_{\mathcal{C}}(n)$. If $\operatorname{End}_{\mathcal{C}}(n)$ is a finite set, as is the case in all of our diagram categories, then we can create a complete "multiplication table". For example, $\operatorname{End}_{\mathcal{S}}(3) = \{1_3, \sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2\}$ where

$\sigma_1 =$  $\sigma_2 =$  $\sigma_3 =$  $\tau_1 =$  $\tau_2 =$ 

Here is the multiplication table for $\mathrm{End}_{\mathcal{S}}(3)$ where the entry in the row labelled by x and column labelled by y is $x \bullet y$:

	1_3	σ_1	σ_2	σ_3	τ_1	τ_2
1_3	1_3	σ_1	σ_2	σ_3	τ_1	τ_2
σ_1	σ_1	1_3	τ_1	τ_2	σ_2	σ_3
σ_2	σ_2	τ_2	1_3	τ_1	σ_3	σ_1
σ_3	σ_3	τ_1	τ_2	1_3	σ_1	σ_2
τ_1	τ_1	σ_3	σ_1	σ_2	τ_2	1_3
τ_2	τ_2	σ_2	σ_3	σ_1	1_3	τ_1

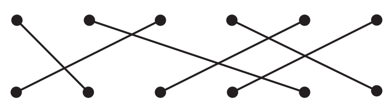
Note: $\mathrm{End}_{\mathcal{S}}(n)$ is often denoted S_n and called a *symmetric group*. With this notation, the table above is the multiplication table for S_3.

Exercise 2.12. Construct multiplication tables for the so-called *diagram monoids** $\mathrm{End}_{\mathcal{P}}(1)$, $\mathrm{End}_{\mathcal{TL}}(2)$, $\mathrm{End}_{\mathcal{S}}(2)$, $\mathrm{End}_{\mathcal{B}}(2)$, and $\mathrm{End}_{\mathcal{TL}}(3)$.

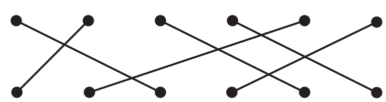
Isomorphisms

A morphism $f : m \to n$ is called an *isomorphism* if there exists a morphism $g : n \to m$ such that $f \circ g = \mathrm{id}_n$ and $g \circ f = \mathrm{id}_m$. Such a g is called an *inverse* of f and we write $g = f^{-1}$.

For example, the partition diagram

$D =$ 

is an isomorphism in the category $\mathcal{P}$ with

$D^{-1} =$  .

Exercise 2.13. Prove that if D is a permutation diagram, then D is an isomorphism in the category $\mathcal{P}$. In particular, given a permutation diagram D, explain how to draw D^{-1}.

Example 2.14. Isomorphisms in **Mat** are invertible matrices. For example, the matrix $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is an isomorphism in **Mat** with $A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$. More generally, any square matrix with nonzero determinant is an isomorphism in **Mat**. The inverse of such a matrix is given by Cramer's formula, and in practice can be computed using elementary row operations.

*A monoid is a set equipped with a single associative operation and an identity element. In any category, $\mathrm{End}_{\mathcal{C}}(n)$ is a monoid. A monoid in which every element is invertible is called a *group*. Every element of $S_n = \mathrm{End}_{\mathcal{S}}(n)$ is invertible (see Exercise 2.13), which is why S_n is called a symmetric group.

Example 2.15. Isomorphisms in **Set** are invertible functions. A standard result in set theory is that a function is invertible if and only if it is a bijection (i.e. one-to-one and onto). Thus, a function is an isomorphism in **Set** exactly when it is a bijection.

A closer look at isomorphisms in $\mathcal{P}$

Our next goal is to prove the following classification of isomorphisms in $\mathcal{P}$:

Proposition 2.16. *D is an isomorphism in $\mathcal{P}$ if and only if D is a permutation diagram.*

Note that proving one of the implications in the previous proposition is Exercise 2.13. Hence, we are required to show that the only isomorphisms in $\mathcal{P}$ are permutation diagrams. To do so, let us first examine the analogous proposition in **Mat**:

Proposition 2.17. *A is an isomorphism in **Mat** if and only if A is a square matrix with nonzero determinant.*

Proof. The fact that square matrices with nonzero determinants are invertible (hence isomorphisms) is discussed in Example 2.14. Here's one way to prove the other direction: The *rank* of a matrix A, denoted $\mathrm{rk}(A)$, is equal to the number of linearly independent columns in A. Here are a couple well-known properties of rank:

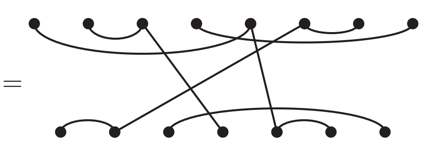
(RK1) $\mathrm{rk}(A) \leq \min\{m, n\}$ whenever A is an $n \times m$ matrix.

(RK2) $\mathrm{rk}(A_2 A_1) \leq \min\{\mathrm{rk}(A_1), \mathrm{rk}(A_2)\}$ for any compatible matrices A_1, A_2.

(RK3) Suppose A is an $n \times n$ matrix. $\det(A) \neq 0$ if and only if $\mathrm{rk}(A) = n$.

Now, suppose A is an $n \times m$ invertible matrix. Then we know there exists an $m \times n$ matrix A^{-1} with $AA^{-1} = I_n$ and $A^{-1}A = I_m$. Thus, using (RK2) we have $n = \mathrm{rk}(I_n) \leq \mathrm{rk}(A)$ and $m = \mathrm{rk}(I_m) \leq \mathrm{rk}(A)$. However, by (RK1) we know $\mathrm{rk}(A) \leq n$ and $\mathrm{rk}(A) \leq m$. It follows that $n = \mathrm{rk}(A) = m$. In particular, A is a square matrix. Moreover, by (RK3) we have $\det(A) \neq 0$. □

We can prove Proposition 2.16 in a similar manner, but we need a gadget to play the role of rank: The *core* of a partition diagram $D : m \to n$, written $\mathrm{core}(D)$, is the number of connected components in D that contain both a top and a bottom vertex. For example, $\mathrm{core}(D) = 3$ where

$D =$ 

The following exercise requests proofs for the core analogs of (RK1)-(RK3):

Exercise 2.18. Prove the following:

(CR1) $\mathrm{core}(D) \le \min\{m, n\}$ whenever $D : m \to n$.

(CR2) $\mathrm{core}(D_2 \bullet D_1) \le \min\{\mathrm{core}(D_1), \mathrm{core}(D_2)\}$ for any compatible partition diagrams D_1, D_2.

(CR3) Suppose D is a partition diagram of type $n \to n$. D is a permutation diagram if and only if $\mathrm{core}(D) = n$.

Exercise 2.19. Using the previous exercise, mimic the proof of Proposition 2.17 to prove Proposition 2.16.

Isomorphic objects

Two objects $m, n \in \mathrm{Ob}\,\mathcal{C}$ are called *isomorphic* in $\mathcal{C}$ if there is an isomorphism in $\mathrm{Hom}_{\mathcal{C}}(m, n)$.

Example 2.20. It follows from Proposition 2.16 that m and n are isomorphic in $\mathcal{P}$ if and only if $m = n$. Similarly, m and n are isomorphism in **Mat** if and only if $m = n$.

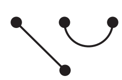
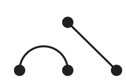
Exercise 2.21. Suppose $\mathcal{D} \subseteq \mathcal{C}$ are categories and $m, n \in \mathrm{Ob}\,\mathcal{D}$. Prove that if m is isomorphic to n in $\mathcal{D}$, then m is isomorphic to n in $\mathcal{C}$.

Exercise 2.22. Suppose $m, n \in \mathbb{Z}_{\ge 0}$ and $\mathcal{D}$ is a subcategory of $\mathcal{P}$. Use Example 2.20 and Exercise 2.21 to show that m and n are isomorphic in $\mathcal{D}$ if and only if $m = n$.

Example 2.23. Since isomorphisms in **Set** are bijections, it follows that two sets are isomorphic in **Set** if and only if they have the same cardinality.

Idempotents

Given a morphism e in any category, the composition $e^2 = e \circ e$ is defined if and only if e is an endomorphism. An *idempotent* is a morphism e such that $e^2 = e$. For example, all identity morphisms are idempotents. However, not all idempotents are identity morphisms. For example, both morphisms in $\mathrm{End}_{\mathcal{P}}(1)$ are idempotents (see Exercise 2.12). Here's another example: Let $B : 1 \to 3$ and $D : 3 \to 1$ denote the following diagrams:

$B =$  $D =$ 

Then $D \bullet B = 1_1$, and one might be tempted to conclude that B and D are inverse isomorphisms, contradicting the fact that 1 and 3 are not isomorphic objects in $\mathcal{P}$ (Example 2.20). Of course, B and D are not inverse isomorphisms since $B \bullet D \neq 1_3$. However, you can check that $B \bullet D$ is an idempotent.

Exercise 2.24. Suppose $f : m \to n$ and $g : n \to m$ are two morphisms in a category such that $g \circ f = \mathrm{id}_m$. Show that $f \circ g$ is an idempotent.

Exercise 2.25. Show that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent in **Mat**. More generally, show that any square matrix whose entries are all 0's except some 1's on the main diagonal is an idempotent in **Mat**. These are not the only idempotents in **Mat**; for instance the matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is an idempotent. In fact any 2×2 matrix with determinant 0 and trace 1 is an idempotent.

Exercise 2.26. Show that constant functions are idempotents in **Set**. Given an example of a function $e : \{1, 2, 3\} \to \{1, 2, 3\}$ that is not a constant function, and not the identity function, but is an idempotent.

Exercise 2.27. Show that the only idempotents which are also isomorphisms are the identity morphisms.

Chapter 3

Functors

3.1 Motivating examples: permutations

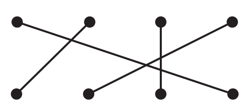
We already know what permutation diagrams are. In this section we will explore permutations in **Mat** (permutation matrices) and in **Set** (set permutations). The connection between permutation diagrams and the latter permutations will serve as motivation for the definition of a functor in §3.2.

Permutation matrices

A *permutation matrix* is a square matrix whose entries are all 0's and 1's such that there is a unique 1 in each row and each column. For example,

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

To each permutation diagram $D : n \to n$, we associate an $n \times n$ permutation matrix $A = F(D)$ as follows: The i, j-entry* of A is 1 if and only if the ith top vertex of D is connected to the jth bottom vertex (reading vertices left to right). For example, the matrix above is associated to the following permutation diagram:



We have just described a map $F : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathbf{Mat}}(n)$ for all $n \in \mathbb{Z}_{\geq 0}$. This map has the property $F(1_n) = I_n$ for all n. Furthermore, it respects the composition in the two categories:

Exercise 3.1. Suppose D and D' are permutation diagrams of type $n \to n$ with associated permutation matrices $A = F(D)$ and $A' = F(D')$. Verify

*The i, j-entry of a matrix is the one in the ith row and jth column.

that $F(D \bullet D') = AA'$. Thus, multiplication of permutation matrices can be accomplished by vertically stacking the corresponding permutation diagrams.

One can also read off properties of a permutation matrix directly from the corresponding permutation diagram. For instance, the *parity* of a permutation diagram D is the parity of the number of crossings when the diagram is drawn in *generic position* (so that exactly two edges meed at each crossing, and edges are not tangent at crossings). For example, the permutation diagram pictured above is even since it has 4 crossings. It turns out that the determinant of a permutation matrix relies only on the parity of the corresponding diagram:

$$\det(F(D)) = (-1)^D = \begin{cases} 0, & \text{if } D \text{ is even;} \\ 1, & \text{if } D \text{ is odd.} \end{cases} \tag{3.1}$$

Set permutations

Given a set X, a *set permutation on* X is a bijection $X \to X$. To each permutation diagram $D : n \to n$ we associate a set permutation $F(D)$ on $[n] = \{1, \ldots, n\}$ as follows: Let $F(D)$ be the function $[n] \to [n]$ defined by mapping $j \mapsto i$ whenever the ith top vertex of D is connected to the jth bottom vertex (reading vertices left to right). For example, if we let D denote the permutation drawn above, then

$$\begin{array}{rccc} F(D): & 1 & \mapsto & 2 \\ & 2 & \mapsto & 4 \\ & 3 & \mapsto & 3 \\ & 4 & \mapsto & 1. \end{array}$$

In this case we have a map $F : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathbf{Set}}([n])$ for all $n \in \mathbb{Z}_{\geq 0}$. This map has some nice properties, similar to the properties we observed in the case of permutation matrices. For instance, $F(1_n) = \mathrm{id}_{[n]}$ for all n.

Exercise 3.2. Suppose D and D' are permutation diagrams of type $n \to n$. Verify that $F(D \bullet D') = F(D) \circ F(D')$. Thus, composition of set permutations reduces to vertically stacking the corresponding permutation diagrams.

3.2 Definition of a functor

In the previous section we saw two examples where morphisms in one category ($\mathcal{S}$) were mapped to morphisms in another category (**Mat** or **Set**), and these mappings sent identity morphisms to identity morphisms and preserved composition. Although it was not emphasized, those maps were consistent on objects in the sense that if D and D' were two diagrams of the same type, then $F(D)$ and $F(D')$ also had the same type (i.e. domain and target). These are the main ingredients in the following definition:

Definition 3.3. Given two categories $\mathcal{C}, \mathcal{D}$, a *functor* F from $\mathcal{C}$ to $\mathcal{D}$, written $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- A map $\mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ written $n \mapsto F(n)$.
- A map $\mathrm{Hom}_{\mathcal{C}}(m,n) \to \mathrm{Hom}_{\mathcal{D}}(F(m),F(n))$, written $g \mapsto F(g)$, for every $m,n \in \mathrm{Ob}\,\mathcal{C}$.

The data above must satisfy the following properties:

(F1) $F(\mathrm{id}_n) = \mathrm{id}_{F(n)}$ for every $n \in \mathrm{Ob}\,\mathcal{C}$.

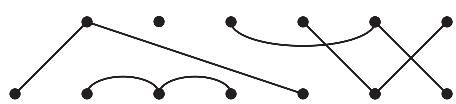
(F2) $F(f \circ g) = F(f) \circ F(g)$ for all morphisms $g : l \to m$ and $f : m \to n$ in $\mathcal{C}$.

Example 3.4. The discussion of permutation matrices in the previous section describes a functor $F : \mathcal{S} \to \mathbf{Mat}$. The functor is given on objects by $F(n) = n$ for all $n \in \mathbb{Z}_{\geq 0}$, and on morphisms by letting $F(D)$ denote the $n \times n$ permutation matrix associated to the permutation diagram $D : n \to n$.

Example 3.5. The discussion of set permutations in the previous section describes a functor $F : \mathcal{S} \to \mathbf{Set}$ given on objects by $F(n) = [n] = \{1,\ldots,n\}$ for all $n \in \mathbb{Z}_{\geq 0}$, and on morphisms by letting $F(D)$ denote the set permutation $[n] \to [n]$ associated to the permutation diagram $D : n \to n$.

Example 3.6. (Identity functors) Suppose $\mathcal{C}$ is an arbitrary category. The functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined to be the identity map on both objects and morphisms. More precisely, $\mathrm{Id}_{\mathcal{C}}(n) = n$ for all $n \in \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Id}_{\mathcal{C}}(g) = g$ for all morphisms $g : m \to n$ in $\mathcal{C}$.

Exercise 3.7. The goal of this exercise is to describe a functor $F : \mathcal{P} \to \mathbf{Rel}$. On objects, this functor is given by $F(n) = [n] = \{1,\ldots,n\}$. To define F on morphisms, let D denote a partition diagram of type $m \to n$. We let $F(D) \subseteq [m] \times [n]$ denote the relation where $(j,i) \in F(D)$ if and only if the ith top vertex of D is connected to the jth bottom vertex (reading vertices left to right). For example, if

$D =$ 

then $F(D) = \{(1,1),(5,1),(6,4),(6,6),(7,3),(7,5)\} \subseteq [7] \times [6]$. Prove that F is a functor.

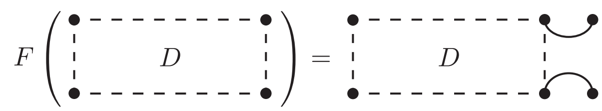
Exercise 3.8. Define $F : \mathcal{NC} \to \mathcal{TL}^{ev}$ on objects by $F(n) = 2n$. For morphisms, F is the bijection

$$\mathrm{Hom}_{\mathcal{NC}}(m,n) \to \mathrm{Hom}_{\mathcal{TL}^{ev}}(2m,2n)$$

prescribed by Exercise 1.12. Verify that this defines a functor $F : \mathcal{NC} \to \mathcal{TL}^{ev}$.

Exercise 3.9. Verify that the following maps define a functor $F : \mathcal{B}^{ev} \to \mathcal{B}^{odd}$. On objects, $F(n) = n+1$. For morphisms $F(D) = D \otimes 1_1$ (i.e. add a single vertical edge on the right).

Exercise 3.10. Consider the following candidate for a functor $F : \mathcal{P} \to \mathcal{P}$. On objects, $F(n) = n+1$. For morphisms, $F(D)$ is the diagram obtained from D by adding a pair of vertices to the right of D, and connecting them to their respective neighbors as illustrated below:



Does F satisfy (F1)? How about (F2)?

3.3 Isomorphic categories

Recall that two sets are called isomorphic if and only if there is a bijection between them (Examples 2.15 and 2.23). This essentially means that isomorphic sets are the "same" set up to relabelling their elements (the bijection prescribes the relabeling). In this section we use functors to define what it means for two categories to be isomorphic, and this definition will mimic that of isomorphic sets. Indeed, roughly speaking, two categories are isomorphic if they are the "same" category up to relabelling all the objects and morphisms. In order to properly discuss isomorphic categories, we first need to generalize the operation of function composition to functors.

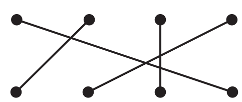
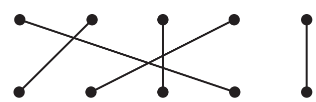
Composition of functors

Since a functor consists of maps (of objects and morphisms) and we can compose maps (of objects and morphisms), we can compose functors. More precisely, given functors $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{C} \to \mathcal{D}$ we define their composition $F \circ G : \mathcal{C} \to \mathcal{E}$ by setting $F \circ G(n) = F(G(n))$ for all objects $n \in \operatorname{Ob}\mathcal{C}$ and $F \circ G(f) = F(G(f))$ for all morphisms f in $\mathcal{C}$. The following computations verify that $F \circ G$ satisfies (F1) and (F2), whence $F \circ G$ is a functor:

$$\begin{aligned}
F \circ G(\mathrm{id}_n) &= F(G(\mathrm{id}_n)) && \text{(definition of } F \circ G) \\
&= F(\mathrm{id}_{G(n)}) && (G \text{ is a functor}) \\
&= \mathrm{id}_{F(G(n))} && (F \text{ is a functor}) \\
&= \mathrm{id}_{F \circ G(n)} && \text{(definition of } F \circ G),
\end{aligned}$$

$$\begin{aligned}
F \circ G(f \circ g) &= F(G(f \circ g)) && \text{(definition of } F \circ G) \\
&= F(G(f) \circ G(g)) && (G \text{ is a functor}) \\
&= F(G(f)) \circ F(G(g)) && (F \text{ is a functor}) \\
&= (F \circ G(f)) \circ (F \circ G(g)) && \text{(definition of } F \circ G).
\end{aligned}$$

Example 3.11. Let $F : \mathcal{S} \to \mathbf{Mat}$ be the functor from Example 3.4 and let $G : \mathcal{S} \to \mathcal{S}$ be the functor given on objects by $G(n) = n + 1$ and on morphisms by $G(D) = D \otimes 1_1$ (add a single vertical edge on the right, as in Example 3.9). The functor $F \circ G : \mathcal{S} \to \mathbf{Mat}$ maps a permutation diagram $D : n \to n$ to $F(G(D)) = F(D \otimes 1_1)$, which is the permutation matrix associated to $D \otimes 1_1$. Here's an explicit example:

 $\mapsto$ 

$$\mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In general, $F \circ G$ maps D to the $(n+1) \times (n+1)$ block matrix $\begin{pmatrix} F(D) & 0 \\ 0 & 1 \end{pmatrix}$.

Exercise 3.12. Let $G : \mathcal{S} \to \mathcal{S}$ be as in Example 3.11. Now, for each $k \in \mathbb{N}$ we define the functor $G^k : \mathcal{S} \to \mathcal{S}$ recursively by setting $G^1 = G$ and $G^k = G \circ G^{k-1}$ for all $k > 1$ (in other words, G^k is obtained by composing G with itself k times). What is $G^k(n)$ for any object $n \in \operatorname{Ob} \mathcal{S}$? Describe $G^k(D)$ for any permutation diagram D as a tensor product of two diagrams. Give a description of $F \circ G^k(D)$ in terms of a block matrix.

Exercise 3.13. Let $F : \mathcal{S} \to \mathbf{Set}$ be as in Example 3.5 and let $G^k : \mathcal{S} \to \mathcal{S}$ be as in Exercise 3.12. Give a detailed description of the functor $F \circ G^k : \mathcal{S} \to \mathbf{Set}$ on both objects and morphisms.

Equality of functors

As we will soon see, to prove two categories are isomorphic amounts to verifying a couple equalities of functors. Let us first be precise about what is required to prove functors are equal. Suppose $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{C}' \to \mathcal{D}'$ are functors. $F = G$ if and only if each of the following hold:

- $\mathcal{C} = \mathcal{C}'$,
- $\mathcal{D} = \mathcal{D}'$,
- $F(n) = F'(n)$ for all $n \in \operatorname{Ob} \mathcal{C}$,
- $F(g) = F'(g)$ for all morphisms g in $\mathcal{C}$.

Example 3.14. For any functor $F : \mathcal{C} \to \mathcal{D}$ we have $F \circ \mathrm{Id}_{\mathcal{C}} = F$. Indeed, both functors are from $\mathcal{C} \to \mathcal{D}$. Moreover, for all $n \in \operatorname{Ob} \mathcal{C}$:

$$\begin{aligned} F \circ \mathrm{Id}_{\mathcal{C}}(n) &= F(\mathrm{Id}_{\mathcal{C}}(n)) && \text{(definition of composition)} \\ &= F(n) && \text{(definition of } \mathrm{Id}_{\mathcal{C}}). \end{aligned}$$

Finally, for all morphisms g in $\mathcal{C}$:

$$\begin{aligned} F \circ \mathrm{Id}_{\mathcal{C}}(g) &= F(\mathrm{Id}_{\mathcal{C}}(g)) && \text{(definition of composition)} \\ &= F(g) && \text{(definition of } \mathrm{Id}_{\mathcal{C}}\text{)}. \end{aligned}$$

Similarly, $\mathrm{Id}_{\mathcal{D}} \circ F = F$.

Exercise 3.15. Let $\mathbf{Rel}_n$ denote the full subcategory of $\mathbf{Rel}$ with

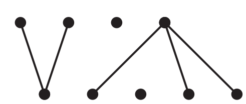
$$\mathrm{Ob}\,\mathbf{Rel}_n = \{[n] \ : \ n \in \mathbb{Z}_{\geq 0}\}.$$

Let $F : \mathcal{P} \to \mathbf{Rel}_n$ denote the functor defined on objects by $F(n) = [n]$. To define F on morphisms, let D denote a partition diagram of type $m \to n$. We let $F(D) \subseteq [m] \times [n]$ denote the relation with $(j, i) \in F(D)$ if and only if the ith top vertex of D is connected to the jth bottom vertex (reading left to right). Explain why F is not equal to the similar functor defined in Example 3.7.

Next, let $G : \mathbf{Rel}_n \to \mathcal{P}$ be the following functor. On objects, $G([n]) = n$. To define G on morphisms, let $R \subseteq [m] \times [n]$ be a relation. Let $G(R)$ denote the partition diagram obtained by drawing an edge connecting the ith top vertex to the jth bottom vertex whenever $(j, i) \in R$. For example, if

$$R = \{(1,1), (1,2), (2,4), (4,4), (5,4)\} \subseteq [5] \times [4],$$

then $G(R)$ is the following diagram:



Prove that $F \circ G = \mathrm{Id}_{\mathbf{Rel}_n}$ but $G \circ F \neq \mathrm{Id}_{\mathcal{P}}$.

Isomorphic categories

Compare the following definitions with the definitions of isomorphisms and isomorphic objects in §2.3[†]. A functor is $F : \mathcal{C} \to \mathcal{D}$ is called an *isomorphism of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{Id}_{\mathcal{D}}$ and $G \circ F = \mathrm{Id}_{\mathcal{C}}$. In this case, we write $G = F^{-1}$. Moreover, we say $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic categories* and write $\mathcal{C} \cong \mathcal{D}$.

Exercise 3.16. Prove the following hold for all categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$:

- $\mathcal{C} \cong \mathcal{C}$.
- If $\mathcal{C} \cong \mathcal{D}$, then $\mathcal{D} \cong \mathcal{C}$.
- If $\mathcal{C} \cong \mathcal{D}$ and $\mathcal{D} \cong \mathcal{E}$, then $\mathcal{C} \cong \mathcal{E}$.

[†]There is a category **Cat** whose objects are categories and morphisms are functors.

Proposition 3.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories whose objects and morphisms form sets*[‡]*. A functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism of categories if and only if the corresponding maps on objects $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ and on morphisms $\operatorname{Hom}_{\mathcal{C}}(m,n) \to \operatorname{Hom}_{\mathcal{D}}(F(m), F(n))$ are all bijections.*

Proof. If F is an isomorphism of categories, then the corresponding maps on objects and morphisms are invertible, hence bijective.

On the other hand, if F induces bijective (hence invertible) maps on objects and morphisms, then the collection of inverse maps prescribes a map $G : \mathcal{D} \to \mathcal{C}$ which is well defined on both objects and morphisms such that $F \circ G = \operatorname{Id}_{\mathcal{D}}$ and $G \circ F = \operatorname{Id}_{\mathcal{C}}$. To complete the proof we need to verify that G is a functor. To do so, notice that

$$\begin{aligned} G(\mathrm{id}_n) &= G(\mathrm{id}_{F\circ G(n)}) && \text{(since } F \circ G = \operatorname{Id}_{\mathcal{D}}\text{)} \\ &= G(F(\mathrm{id}_{G(n)})) && \text{(since } F \text{ is a functor)} \\ &= \mathrm{id}_{G(n)} && \text{(since } G \circ F = \operatorname{Id}_{\mathcal{C}}\text{)} \end{aligned}$$

whence G satisfies (F1). The following verifies that G satisfies (F2):

$$\begin{aligned} G(f \circ g) &= G(F(G(f)) \circ F(G(g))) && \text{(since } F \circ G = \operatorname{Id}_{\mathcal{D}}\text{)} \\ &= G(F(G(f) \circ G(g))) && \text{(since } F \text{ is a functor)} \\ &= G(f) \circ G(g) && \text{(since } G \circ F = \operatorname{Id}_{\mathcal{C}}\text{)} \end{aligned}$$

□

Example 3.18. Let $\mathbf{Bij}_n$ denote the subcategory of $\mathbf{Set}$ defined by

$$\operatorname{Ob}\mathbf{Bij}_n = \{[n] \ : \ n \in \mathbb{Z}_{\geq 0}\},$$

$$\operatorname{Hom}_{\mathbf{Bij}_n}([m],[n]) = \{\text{bijections from } [m] \to [n]\}.$$

Since there's a bijection from $[m]$ to $[n]$ if and only if $m = n$, it follows that $\operatorname{Hom}_{\mathbf{Bij}_n}([m],[n]) = \varnothing$ whenever $m \neq n$. Therefore, morphisms in $\mathbf{Bij}_n$ are exactly set permutations. Let $F : \mathcal{S} \to \mathbf{Bij}_n$ be the functor which agrees with the functor of the same name from Example 3.5 on everything except their target categories. Since the assignment $D \mapsto F(D)$ gives a bijection between diagram permutations of type $n \to n$ and set permutations on $[n]$, it follows from Proposition 3.17 that F is an isomorphism of categories. Hence $\mathcal{S} \cong \mathbf{Bij}_n$.

Exercise 3.19. Let $\mathbf{PMat}$ denote the subcategory of $\mathbf{Mat}$ defined by setting $\operatorname{Ob}\mathbf{PMat} = \operatorname{Ob}\mathbf{Mat}$ and $\operatorname{Hom}_{\mathbf{PMat}}(m,n) = \{n \times m \text{ permutation matrices}\}$ for all $m, n \in \mathbb{Z}_{\geq 0}$. Prove that $\mathcal{S} \cong \mathbf{PMat}$.

[‡]Such categories are called *small*. All of the diagram categories we will consider are small, but there are plenty of *large* (not small) categories. For example **Set** is large since the collection of all sets does not form a set – just ask Bertrand Russell. I expect that the proposition is true without the assumption that $\mathcal{C}$ and $\mathcal{D}$ are small, but I don't care enough to check the details. In any case, I'd rather only talk about bijections between sets as opposed to ones between proper classes.

Exercise 3.20. Prove that $\mathcal{NC} \cong \mathcal{TL}^{ev}$.

Exercise 3.21. Why is the functor $F : \mathcal{P} \to \mathbf{Rel}_n$ in Exercise 3.15 is not an isomorphism. Explain why knowing F is not an isomorphism is not sufficient to conclude that $\mathcal{P}$ and $\mathbf{Rel}_n$ are not isomorphic. Now, count the number of elements (relations) in the set $\mathrm{Hom}_{\mathbf{Rel}_n}([2],[1])$. Use that count along with Exercise 1.1 and Proposition 3.17 to prove that $\mathcal{P}$ and $\mathbf{Rel}_n$ are not isomorphic.

Chapter 4

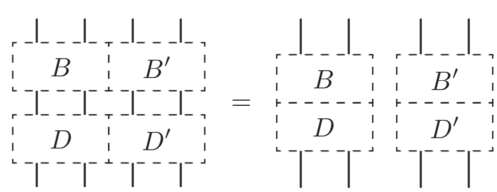
Strict monoidal categories

4.1 Motivating examples: other matrix and function operations

Up to this point, when discussing a category we have focussed on a single operation (vertical stacking of diagram, matrix multiplication, function composition, etc.). In this chapter we develop the categorical framework which allows for a pair of operations. The main example to keep in mind is a diagram category with the two operations of vertical and horizontal stacking. There are a couple important properties of this pair of operations. First, horizontally stacking identity diagrams yields another identity diagram:

$$1_n \otimes 1_{n'} = 1_{n+n'}. \tag{4.1}$$

Also, let us make the following diagrammatic observation:



In other words,

$$(B \otimes B') \bullet (D \otimes D') = (B \bullet D) \otimes (B' \bullet D') \tag{4.2}$$

for all diagrams $B : m \to n$, $B' : m' \to n'$, $D : l \to m$, $D' : l' \to m'$. In this section we look at a few more examples of operations in categories which satisfy formulae similar to (4.1) and (4.2).

Example 4.1. (Direct sums of matrices) Given an $n \times m$ matrix A and a $n' \times m'$ matrix A', their *direct sum* is the $(n + n') \times (m + m')$ block diagonal matrix

$$A \oplus A' = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}.$$

The direct sum of identity matrices is another identity matrix: $I_n \oplus I_{n'} = I_{n+n'}$. Moreover, given matrices A $(n \times m)$, A' $(n' \times m')$, B $(m \times l)$, and B' $(m' \times l')$ we have

$$(A \oplus A')(B \oplus B') = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & A'B' \end{pmatrix} = (AB) \oplus (A'B').$$

Example 4.2. (Kronecker products of matrices) Given an $n \times m$ matrix A and a $n' \times m'$ matrix A', their *Kronecker product* is the $(nn') \times (mm')$ matrix which has the following block form:

$$A \otimes A' = \begin{pmatrix} a_{1,1}A' & \cdots & a_{1,m}A' \\ \vdots & & \vdots \\ a_{n,1}A' & \cdots & a_{n,m}A' \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix}.$$

For example,

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 4 & 2 \\ -1 & 0 & 2 & 0 \\ -3 & 2 & 6 & -4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 \end{pmatrix}. \tag{4.3}$$

Kronecker products of identity matrices are identity matrices: $I_n \otimes I_{n'} = I_{nn'}$.

Exercise 4.3. Given matrices A $(n \times m)$, A' $(n' \times m')$, B $(m \times l)$, and B' $(m' \times l')$ prove that

$$(A \otimes A')(B \otimes B') = (AB) \otimes (A'B'). \tag{4.4}$$

Hint: We can index the entry $a_{i,j}a'_{i',j'}$ in $A \otimes A'$ by a tuple (i, i', j, j'). For example, the $(1, 3, 2, 1)$-entry in (4.3) is 6. Compute the (i, i', j, j')-entry in both sides of (4.4).

Example 4.4. (Cartesian products) Given functions $f : M \to N$ and $f' : M' \to N'$, their *Cartesian product* is the function

$$\begin{aligned} f \times f' : M \times M' &\to N \times N' \\ (x, x') &\mapsto (f(x), f'(x')). \end{aligned}$$

The Cartesian product of two identity functions is another identity function: $\mathrm{id}_N \times \mathrm{id}_{N'} = \mathrm{id}_{N \times N'}$. Moreover, given functions $f : M \to N$, $f' : M' \to N'$, $g : L \to M$, and $g' : L' \to M'$, we have

$$\begin{aligned} (f \times f') \circ (g \times g')(x, x') &= f \times f'(g \times g'(x, x')) \\ &= f \times f'(g(x), g'(x')) \\ &= (f(g(x)), f'(g'(x'))) \end{aligned}$$

$$\begin{aligned} &= (f \circ g(x), f' \circ g'(x')) \\ &= (f \circ g) \times (f' \circ g')(x, x') \end{aligned}$$

for all $(x, x') \in L \times L'$. Therefore $(f \times f') \circ (g \times g') = (f \circ g) \times (f' \circ g')$.

Example 4.5. (Disjoint unions) The *disjoint union of sets* M and M' is the set

$$M \sqcup M' = \{(x, 0) \ : \ x \in M\} \cup \{(x', 1) \ : \ x' \in M'\}.$$

Now, given functions $f : M \to N$ and $f' : M' \to N'$, their *disjoint union* is the function

$$\begin{array}{rcl} f \sqcup f' : M \sqcup M' & \to & N \sqcup N' \\ (x, 0) & \mapsto & (f(x), 0) \\ (x', 1) & \mapsto & (f'(x'), 1) \end{array}$$

Note that the disjoint unions of two identity functions is another identity function: $\mathrm{id}_N \sqcup \mathrm{id}_{N'} = \mathrm{id}_{N \sqcup N'}$.

Exercise 4.6. Given functions $f : M \to N$, $f' : M' \to N'$, $g : L \to M$, and $g' : L' \to M'$, verify that $(f \sqcup f') \circ (g \sqcup g') = (f \circ g) \sqcup (f' \circ g')$ by calculating the images of $(x, 0)$ and $(x', 1)$ for arbitrary $x \in L$ and $x' \in L'$.

4.2 Bifunctors

Compare the following definition with the operation of horizontal stacking in diagram categories and the other examples of operations in §4.1.

Definition 4.7. A *bifunctor* $\odot$ on a category $\mathcal{C}$ consists of the following data:

- A map $\mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$ written $(n, n') \mapsto n \odot n'$.
- A map $\mathrm{Hom}_{\mathcal{C}}(m, n) \times \mathrm{Hom}_{\mathcal{C}}(m', n') \to \mathrm{Hom}_{\mathcal{C}}(n \odot n', m \odot m')$, written $(f, f') \mapsto f \odot f'$, for every $m, n, m', n' \in \mathrm{Ob}\,\mathcal{C}$.

The data above must satisfy the following properties:

(B1) $\mathrm{id}_n \odot \mathrm{id}_{n'} = \mathrm{id}_{n \odot n'}$ for every $n, n' \in \mathrm{Ob}\,\mathcal{C}$.

(B2) $(f \odot f') \circ (g \odot g') = (f \circ g) \odot (f' \circ g')$ for all morphisms $f : m \to n$, $f' : m' \to n'$, $g : l \to m$, $g' : l' \to m'$ in $\mathcal{C}$.

Example 4.8. Horizontal stacking gives a bifunctor $\otimes$ on $\mathcal{P}$. We already know how this bifunctor acts on morphisms (stack diagrams horizontally). On objects we set $n \otimes n' = n + n'$. Properties (4.1) and (4.2) verify that $\otimes$ satisfies (B1) and (B2) respectively. Note that $\otimes$ can also be viewed as a bifunctor on $\mathcal{NC}$, $\mathcal{B}$, $\mathcal{TL}$, or $\mathcal{S}$ (see Exercise 1.18).

Exercise 4.9. Determine which of the following diagram categories horizontal stacking defines a bifunctor $\otimes$ on: $\mathcal{B}^{ev}$, $\mathcal{B}^{odd}$, $\mathcal{TL}^{ev}$, $\mathcal{TL}^{odd}$.

Example 4.10. The direct sum of matrices gives a bifunctor $\oplus$ on **Mat**. On objects we set $n \oplus n' = n + n'$. The definition of $\oplus$ on morphisms (matrices) as well as the verification of (B1) and (B2) can be found in Example 4.1.

Example 4.11. The Kronecker product of matrices gives another bifunctor $\otimes$ on **Mat**. On objects we set $n \otimes n' = nn'$. The definition of $\otimes$ on morphisms (matrices) as well as the verification of (B1) and (B2) can be found in Example 4.2 and Exercise 4.3.

Example 4.12. It follows from Example 4.4 that the Cartesian product defines a bifunctor $\times$ on **Set**.

Example 4.13. It follows from Example 4.5 and Exercise 4.6 that the disjoint union defines a bifunctor $\sqcup$ on **Set**.

4.3 Definition of a strict monoidal category

Our main example of a bifunctor (horizontal stacking of diagrams) satisfies some additional nice properties. First off, horizontal stacking is associative:

$$(D_1 \otimes D_2) \otimes D_3 = D_1 \otimes (D_2 \otimes D_3) \tag{4.5}$$

for all partition diagrams D_1, D_2, D_3. Moreover, there is a special (empty) diagram that acts as a unit with respect to horizontal stacking:

$$D \otimes 1_0 = D = 1_0 \otimes D \tag{4.6}$$

for any partition diagram D. A category equipped with a bifunctor and such nice properties gets a special name:

Definition 4.14. A *strict monoidal category* is a triple of data $(\mathcal{C}, \odot, \mathbb{1})$ where:

- $\mathcal{C}$ is a category.
- $\odot$ is a bifunctor on $\mathcal{C}$.
- $\mathbb{1}$ is an object in $\mathcal{C}$ called the *unit object*.

The data above must satisfy the following properties:

(M1) $f \odot \mathrm{id}_{\mathbb{1}} = f = \mathrm{id}_{\mathbb{1}} \odot f$ for every morphism f in $\mathcal{C}$.

(M2) $(f \odot g) \odot h = f \odot (g \odot h)$ for all morphisms f, g, h in $\mathcal{C}$.

Example 4.15. Formulae (4.5) and (4.6) verify that $(\mathcal{P}, \otimes, 0)$ is a strict monoidal category. Similarly, $(\mathcal{NC}, \otimes, 0)$, $(\mathcal{B}, \otimes, 0)$, $(\mathcal{TL}, \otimes, 0)$, and $(\mathcal{S}, \otimes, 0)$ are all strict monoidal categories.

Example 4.16. Direct sums of matrices (see Example 4.1) give rise to a strict monoidal category $(\mathbf{Mat}, \oplus, 0)$. The following computation verifies (M1):

$$A \oplus 0 = A = 0 \oplus A,$$

where 0 in the line above denotes the unique 0×0 matrix, which exists by convention (see Remark 2.4*). Here's the verification of (M2):

$$(A \oplus B) \oplus C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \oplus C = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} = A \oplus \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} = A \oplus (B \oplus C).$$

Exercise 4.17. Show that $(\mathbf{Mat}, \otimes, 1)$ is a strict monoidal category. Take care in verifying (M2).

Exercise 4.18. Show that $A \oplus B$ and $A \otimes B$ are permutation matrices whenever A and B are. Conclude that both $(\mathbf{PMat}, \oplus, 0)$ and $(\mathbf{PMat}, \otimes, 1)$ are strict monoidal categories.

Exercise 4.19. Although the defining properties of a strict monoidal category are in terms of only morphisms, the analogous properties for objects hold too. Apply axiom (C1) to (M1) and (M2) to prove the following hold in any strict monoidal category:

1. $n \odot \mathbb{1} = n = \mathbb{1} \odot n$ for all $n \in \mathrm{Ob}\,\mathcal{C}$.
2. $(l \odot m) \odot n = l \odot (m \odot n)$ for all $l, m, n \in \mathrm{Ob}\,\mathcal{C}$.

As a consequence, we have the following:

Proposition 4.20. *Unit objects in strict monoidal categories are unique. In fact, if $(\mathcal{C}, \odot, \mathbb{1})$ is a strict monoidal category, and $\mathbb{1}'$ is any object in $\mathcal{C}$ satisfying either $\mathbb{1}' \odot n = n$ or $n \odot \mathbb{1}' = n$ for all $n \in \mathrm{Ob}\,\mathcal{C}$, then $\mathbb{1}' = \mathbb{1}$.*

Proof. By part 1 of Exercise 4.19 we have $\mathbb{1}' \odot \mathbb{1} = \mathbb{1}' = \mathbb{1} \odot \mathbb{1}'$. The assumption on $\mathbb{1}'$ implies that either $\mathbb{1}' \odot \mathbb{1} = \mathbb{1}$ or $\mathbb{1} \odot \mathbb{1}' = \mathbb{1}$. □

4.4 Non-strict monoidal categories

Not every bifunctor determines a strict monoidal category. For example, consider the bifunctor $\times$ on **Set**. If we're careful, we can see that this bifunctor is not associative on objects:

$$(L \times M) \times N \neq L \times (M \times N). \tag{4.7}$$

Indeed, elements of the left side have the form $((x, y), z)$ whereas elements of the right look like $(x, (y, z))$. These are close to the same, but they are not

*That remark used to be a footnote so you may have ignored it, which is fine until now.

equal. It follows from part 2 of Exercise 4.19 that $\times$ cannot be used to give **Set** the structure of a strict monoidal category.

We can also arrive at the same conclusion by looking for unit objects. Suppose $\mathbb{1}$ is a set such that $N \times \mathbb{1} = N$ for any set N. Comparing sizes will lead you to the conclusion that $\mathbb{1}$ must be a singleton set: $\mathbb{1} = \{\star\}$. Although $N \times \{\star\}$ is not equal to N on the nose, there is an obvious way to identify their elements: $x \leftrightarrow (x, \star)$. But what singleton set do you pick? Indeed, there is not a unique choice for a unit object. Hence by Proposition 4.20, $\times$ cannot be used to give **Set** the structure of a strict monoidal category.

The triple $(\mathbf{Set}, \times, \{\star\})$ is an example of a *non-strict monoidal category.* I don't want to give the precise definition of that term since our primary concern are diagram categories, which are strict monoidal categories. However, I think it's useful to know that there are plenty of important bifunctors in mathematics which are "almost associative" and "almost" have unit objects. Making these "almost-statements" precise involves plenty of isomorphisms. Indeed, although the sets in (4.7) are not equal, they are isomorphic in **Set** via the function $((x, y), z) \mapsto (x, (y, z))$. Moreover, although there are infinitely many singleton sets, they are all isomorphic in **Set**.

Exercise 4.21. Show that the bifunctor $\sqcup$ on **Set** is not associative. Find a unit object $\mathbb{1}$ so that $(\mathbf{Set}, \sqcup, \mathbb{1})$ is a non-strict monoidal category, whatever that means. Is the unit object unique?

Strictification

One reason I am okay with dodging the definition of a non-strict monoidal category is that it turns out every non-strict monoidal category is equivalent (in some precise way) to a strict one. The process of replacing a non-strict monoidal category with an equivalent strict monoidal category is called *strictification.* The rest of this section is devoted to the strictifications of $(\mathbf{FinSet}, \sqcup, \mathbb{1})$ and $(\mathbf{FinSet}, \times, \{\star\})$, where **FinSet** is the full subcategory of **Set** whose objects are finite sets. To start, let $\mathbf{Set}_n$ denote the full subcategory of **Set** with

$$\operatorname{Ob}\mathbf{Set}_n = \{[n] \ : \ n \in \mathbb{Z}_{\geq 0}\}.$$

Exercise 4.22. Show that neither the disjoint union $\sqcup$ nor the Cartesian product $\times$ define a bifunctor on $\mathbf{Set}_n$.

To strictify the disjoint union, we define the bifunctor $\uplus$ on the category $\mathbf{Set}_n$ as follows. On objects we set $[n] \uplus [n'] = [n + n']$. To define $\uplus$ on morphisms, suppose $f : [m] \to [n]$ and $g : [m'] \to [n']$ are two functions. Set

$$\begin{array}{rcll} f \uplus g : [m + m'] & \to & [n + n'] & \\ j & \mapsto & f(j) & (1 \leq j \leq m) \\ m + j & \mapsto & n + g(j) & (1 \leq j \leq m') \end{array}$$

Let's compare the operations $\sqcup$ and $\uplus$ with a specific example. Let f and g denote the following functions:

$$\begin{tikzcd}[row sep=tiny]
f : [3] \arrow[r] & {[2]} \\
1 \arrow[dr] & 1 \\
2 \arrow[ur] & 2 \\
3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny]
g : [3] \arrow[r] & {[3]} \\
1 \arrow[dr] & 1 \\
2 \arrow[dr] & 2 \\
3 \arrow[uur] & 3
\end{tikzcd}$$

Applying the operations $\sqcup$ and $\uplus$ gives the following:

$$\begin{tikzcd}[row sep=tiny]
f \sqcup g : [3] \sqcup [3] \arrow[r] & {[2] \sqcup [3]} \\
(1,0) \arrow[dr] & (1,0) \\
(2,0) \arrow[ur] & (2,0) \\
(3,0) \arrow[ur] & \\
(1,1) \arrow[dr] & (1,1) \\
(2,1) \arrow[dr] & (2,1) \\
(3,1) \arrow[uur] & (3,1)
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny]
f \uplus g : [6] \arrow[r] & {[5]} \\
1 \arrow[dr] & 1 \\
2 \arrow[ur] & 2 \\
3 \arrow[ur] & \\
4 \arrow[dr] & 3 \\
5 \arrow[dr] & 4 \\
6 \arrow[uur] & 5
\end{tikzcd}$$

Both operations are, in some sense, obtained by stacking the rule for f on top of the rule for g. The only difference is the labelling of the elements in the domain and target.

Exercise 4.23. Show that $\uplus$ satisfies (B1) and (B2). Find the unique unit object $\mathbb{1} \in \operatorname{Ob} \mathbf{Set}_n$ that makes $(\mathbf{Set}_n, \uplus, \mathbb{1})$ a strict monoidal category. Be sure to verify (M1) and (M2).

Exercise 4.24. Show that $f \uplus g$ is a bijection whenever f and g are. Conclude that $(\mathbf{Bij}_n, \uplus, \mathbb{1})$ is also a strict monoidal category.

To strictify the Cartesian product, we will define a bifunctor $*$ on $\mathbf{Set}_n$. For objects we set $[n] * [n'] = [nn']$. Note that $[n] \times [n']$ and $[n] * [n']$ are isomorphic in $\mathbf{Set}$. My favorite bijection between those sets is the following:

$$\begin{aligned} \phi_{n,n'} : [n] \times [n'] &\to [nn'] \\ (i,j) &\mapsto j + (i-1)n' \end{aligned}$$

For example, the following illustrates $\phi_{3,4} : [3] \times [4] \to [12]$:

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \end{array} \xrightarrow{\phi_{3,4}} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}$$

Now, we define $*$ on morphisms by setting

$$f * g = \phi_{n,n'} \circ (f \times g) \circ \phi_{m,m'}^{-1} \tag{4.8}$$

whenever $f : [m] \to [n]$ and $g : [m'] \to [n']$. For example, let $f : [3] \to [2]$ and $g : [3] \to [3]$ be the functions defined above. The map $f * g : [3] * [3] \to [2] * [3]$ can be found using the following maps between arrays:

$$\begin{array}{ccc}
\begin{matrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{matrix} & \xrightarrow{f \times g} & \begin{matrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \end{matrix} \\
{\scriptstyle \phi_{3,3}^{-1}} \uparrow & & {\scriptstyle \phi_{2,3}} \downarrow \\
\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix} & \xrightarrow{f * g} & \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}
\end{array}$$

For instance, $f * g(7) = 5$ since $7 \xrightarrow{\phi_{3,3}^{-1}} (3,1) \xrightarrow{f \times g} (2,2) \xrightarrow{\phi_{2,3}} 5$.

Exercise 4.25. Use (4.8) to show that $*$ satisfies (B1) and (B2).

Exercise 4.26. Given arbitrary functions $f : [m] \to [n]$ and $g : [m'] \to [n']$, find an explicit formula for $f * g(k)$ for any $1 \leq k \leq mm'$.
[Hint: Write $k = j + (i-1)m'$ with $1 \leq j \leq m'$ and $1 \leq i \leq m$.]

Exercise 4.27. Prove that $(\mathbf{Set}_n, *, [1])$ is a strict monoidal category.

Exercise 4.28. Show that $f * g$ is a bijection whenever f and g are. Conclude that $(\mathbf{Bij}_n, *, [1])$ is also a strict monoidal category.

4.5 Strict monoidal functors

In Chapter 3 we saw how functors give us a precise way to compare vertical stacking in diagram categories with operations such as matrix multiplication and function composition. In order to simultaneously compare vertical horizontal stacking of diagram categories with the multiple operations in other strict monoidal categories, our functors need to satisfy some additional properties to be sure that monoidal structure is preserved.

Definition 4.29. Given two strict monoidal categories $(\mathcal{C}, \odot, \mathbb{1})$ and $(\mathcal{D}, \odot, \mathbb{1})$, a *strict monoidal functor* from $(\mathcal{C}, \odot, \mathbb{1})$ to $(\mathcal{D}, \odot, \mathbb{1})$ is a functor $F : \mathcal{C} \to \mathcal{D}$ that satisfies the following properties:

(MF1) $F(\mathbb{1}) = \mathbb{1}$.
(MF2) $F(f \odot g) = F(f) \odot F(g)$ for all morphisms f and g in $\mathcal{C}$.

Example 4.30. Consider the functor $F : \mathcal{S} \to \mathbf{Mat}$ that maps each permutation diagram to its corresponding permutation matrix (see Example 3.4). We will show F is a strict monoidal functor from $(\mathcal{S}, \otimes, 0)$ to $(\mathbf{Mat}, \oplus, 0)$. Since $F(0) = 0$, it follows that (MF1) holds. You are encouraged to fill in the details in the following verification of (MF2):

$$F(D \otimes D') = \begin{pmatrix} F(D) & 0 \\ 0 & F(D') \end{pmatrix} = F(D) \oplus F(D').$$

Exercise 4.31. Show that the functor $F : \mathcal{S} \to \mathbf{Set}_n$ that maps each permutation diagram to its corresponding set permutation is a strict monoidal functor from $(\mathcal{S}, \otimes, 0)$ to $(\mathbf{Set}_n, \uplus, \mathbb{1})$.

Exercise 4.32. Explain how to define a bifunctor $\uplus$ on $\mathbf{Rel}_n$ so that functor $F : \mathcal{P} \to \mathbf{Rel}_n$ from Exercise 3.15 is a strict monoidal functor from $(\mathcal{P}, \otimes, 0)$ to $(\mathbf{Rel}_n, \uplus, \mathbb{1})$. Of course, you need to determine the unit object too.

Exercise 4.33. Prove that $F \circ G$ is a strict monoidal functor whenever F and G are.

Isomorphic strict monoidal categories

If there is a strict monoidal functor from $(\mathcal{C}, \odot, \mathbb{1})$ to $(\mathcal{D}, \odot, \mathbb{1})$ which is also an isomorphism of categories, then we say that $(\mathcal{C}, \odot, \mathbb{1})$ and $(\mathcal{D}, \odot, \mathbb{1})$ are *isomorphic*, or that $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic as strict monoidal categories* if the monoidal structure is understood from the context.

Example 4.34. The strict monoidal categories $(\mathbf{PMat}, \oplus, 0)$ and $(\mathcal{S}, \otimes, 0)$ are isomorphic. Indeed, arguing as in Example 4.30, the functor $F : \mathcal{S} \to \mathbf{PMat}$ which maps a permutation diagram to its corresponding permutation matrix is a strict monoidal functor. Moreover, F is an isomorphism of categories (see Exercise 3.19).

Exercise 4.35. Prove that $(\mathbf{Bij}_n, \uplus, \mathbb{1})$ is isomorphic to $(\mathcal{S}, \otimes, 0)$. Use Example 4.34 along with Exercises 3.16 and 4.33 to prove that $(\mathbf{Bij}_n, \uplus, \mathbb{1})$ and $(\mathbf{PMat}, \oplus, 0)$ are isomorphic.

Exercise 4.36. Show $\mathcal{NC}$ and $\mathcal{TL}^{ev}$ are isomorphic as strict monoidal categories.

Exercise 4.37. Prove that $(\mathbf{PMat}, \otimes, 1)$ is isomorphic to $(\mathbf{Bij}_n, *, [1])$.

Exercise 4.38. Give a diagrammatic description of a bifunctor $\boxtimes$ on $\mathcal{S}$ such that $(\mathcal{S}, \boxtimes, 1)$ is a strict monoidal category which is isomorphic to $(\mathbf{PMat}, \otimes, 1)$ and $(\mathbf{Bij}_n, *, [1])$.

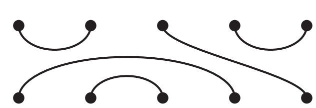
Chapter 5

Presentations of categories by generators and relations

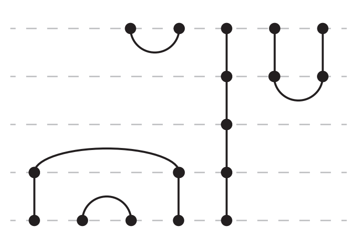
5.1 Motivating example: $\mathcal{TL}$

In this chapter we will describe various diagram categories $\mathcal{D}$ by so-called "generators and relations". The "generators" in such a description are diagrams such that all other diagrams in $\mathcal{D}$ can be built by stacking generators and identity morphisms, both vertically and horizontally. For example, in $\mathcal{TL}$ the standard generators are the cup and the cap:

It is not difficult to see that every other Temperley-Lieb diagram can be constructed by stacking identity diagrams, cups, and caps. In fact, every Temperley-Lieb diagram can be decomposed as a vertical stack of levels, where each level consists of at most one cup/cap along with some vertical strands (identity morphisms). For example, the diagram

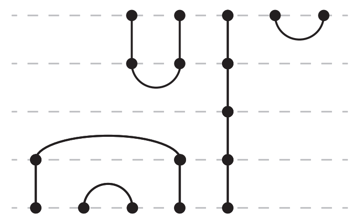


can be decomposed as follows:



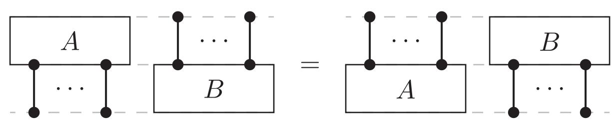
(5.1)

Of course, there is not a unique way to decompose a diagram. For example, the diagram above could have also been decomposed as follows:



(5.2)

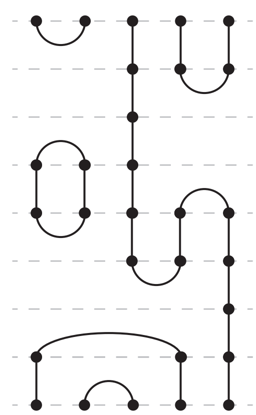
The “relations” correspond to the “diagrammatic moves” necessary to verify two different decompositions correspond to the same diagram. For example, the decompositions in (5.1) and (5.2) are related by a *sliding relation*:



Sliding relations hold in every stitch monoidal category since

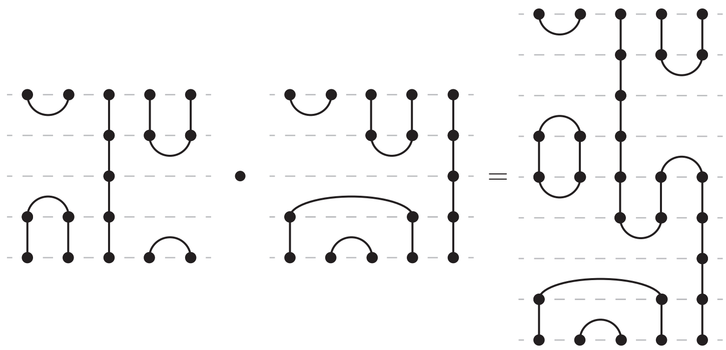
$$(A \otimes 1_{n'}) \circ (1_m \otimes B) = A \otimes B = (1_n \otimes B) \circ (A \otimes 1_{m'})$$

by (C2) and (B2). There are, however, different decompositions of the same diagram which cannot be related by only sliding relations. For example, here is another, more complicated way to decompose that same diagram:

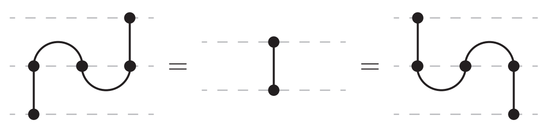


(5.3)

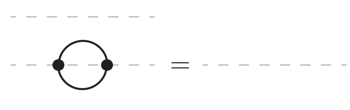
This last decomposition probably wouldn’t be your first choice when decomposing the diagram, but it could arise when stacking two other diagrams which are already decomposed:



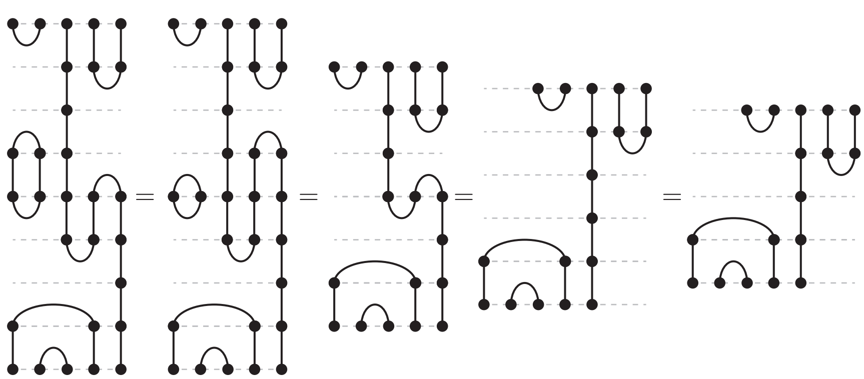
The fundamental relations in $\mathcal{TL}$ that will allow us to move between the decompositions (5.3) and (5.1) are the *straightening relations*:



and the *loop relation*:

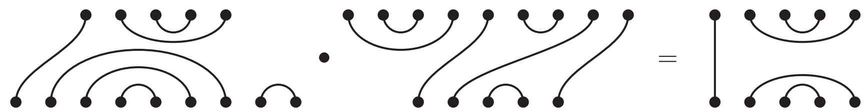


The following shows how to get from (5.3) to (5.1).

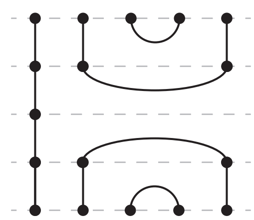


The first equality uses a sliding relation, the second a loop relation, the third a straightening relation. The last equality comes from a relation that says that you can remove any level which is corresponds to an identity morphism, which follows from (C2).

Exercise 5.1. Consider the following equation that holds in $\mathcal{TL}$:



Construct decompositions of the two diagrams on the left side of the equality, stack them accordingly, and use the relations described above to simplify the stack until it looks like the following decomposition of the diagram on the right:



Each step in your simplification should modify at most two levels of the decomposition.

While it's easy to see that the relations described above hold in $\mathcal{TL}$, what's not obvious is that these relations are enough to completely determine whether or not two diagrams are equal in $\mathcal{TL}$. To be a bit more precise, two decompositions of Temperley-Lieb diagrams correspond to the same morphism in $\mathcal{TL}$ if and only if one can be transformed to the other by a sequence of straightening relations, loop relations, and other relations that hold in all strict monoidal categories (such as sliding relations). We will prove this fact in §5.4 (see Theorem 5.26). Afterwards, we will consider generators and relations for other diagram categories. First, in §5.2 and §5.3 we will develop the theory of free strict monoidal categories given by generators and relations. This will allow us to prove (and state precisely) theorems which describe diagram categories by generators and relations. Essentially, we will develop the tools to construct a strict monoidal category $\mathcal{C}$ in which morphisms are related *only* by our candidates for the relations that describe a diagram category $\mathcal{D}$. Then we can prove $\mathcal{D}$ is given by our candidates for generators and relations by showing that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic as strict monoidal categories.

5.2 Free monoidal categories

Quivers

A *quiver*, Q, consists of a set of *vertices* V, a set of *arrows* A, and a function $A \to V \times V$ which assigns each arrow an initial vertex and a terminal vertex. All quivers in these notes will have $V = \mathbb{Z}_{\geq 0}$. With this in mind, our quivers are collections of arrows between nonnegative integers. For example, we write

$$Q = \{a : 0 \to 1, \quad b : 1 \to 0, \quad c : 0 \to 2, \quad d : 2 \to 0\}$$

for the quiver Q with $A = \{a, b, c, d\}$ and

$$\begin{array}{ccc} A & \to & \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \\ a & \mapsto & (0,1) \\ b & \mapsto & (1,0) \\ c & \mapsto & (0,2) \\ d & \mapsto & (2,0). \end{array}$$

Quivers also go by the name of *directed graphs* and have applications to multiple areas of mathematics. For us, they are merely a nice setting to discuss morphism-like things (arrows) which do not necessarily belong to a category. In fact, arrows in our quivers should be viewed merely as formal symbols (as opposed to diagrams, matrices, functions, etc.). What is $a : 0 \to 1$? It's an arrow in a quiver. What's a quiver? It's a collection of arrows.

Formal combinations of arrows

Given a quiver Q, we define a sequence of quivers $Q_0, Q_1, Q_2, \ldots$ inductively as follows. First, let Q_0 denote the quiver which contains all the arrows in Q as well as an arrow $1_n : n \to n$ for each $n \in \mathbb{Z}_{\geq 0}$. Now, for each $i \in \mathbb{Z}_{\geq 0}$ we let Q_{i+1} denote the quiver obtained from Q_i by adding new arrows in one of the following two ways:
(1) Whenever $a : m \to n$ and $b : l \to m$ are arrows in Q_i, at least one of which is not in Q_{i-1}, we add a new arrow $a \circ b : l \to n$ to Q_{i+1}.
(2) Whenever $a : m \to n$ and $a' : m' \to n'$ are arrows in Q_i, at least one of which is not in Q_{i-1}, we add a new arrow $a \odot a' : m + m' \to n + n'$ to Q_{i+1}.

For example, suppose Q consists of a single arrow $a : 0 \to 1$. Then Q_0 consists of the following arrows:

$$a : 0 \to 1, \quad 1_0 : 0 \to 0, \quad 1_1 : 1 \to 1, \quad 1_2 : 2 \to 2, \ldots$$

Q_1 consists of all the arrows above along with the following arrows:

$$a \circ 1_0 : 0 \to 1, \quad 1_1 \circ a : 0 \to 1, \quad 1_n \circ 1_n : n \to n \quad (n \in \mathbb{Z}_{\geq 0}),$$

$$a \odot a : 0 \to 2, \quad a \odot 1_n : n \to n+1, \quad 1_n \odot a : n \to n+1, \quad 1_m \odot 1_n : m+n \to m+n.$$

Note that the arrows listed above are all distinct. For example, a and $a \circ 1_0$ are different arrows in Q_1. We'll get to simplifying these expressions soon. For now, we are only formally applying the operations $\circ$ and $\odot$, whence the quivers Q_i grow quickly. For instance, even if we restrict our attention to diagrams of type $0 \to 0$, there are 11 distinct arrows in Q_2:

$$\begin{array}{cccc} 1_0, & 1_0 \circ 1_0, & 1_0 \odot 1_0, & \\ 1_0 \circ (1_0 \circ 1_0), & (1_0 \circ 1_0) \circ 1_0, & 1_0 \odot (1_0 \circ 1_0), & (1_0 \odot 1_0) \circ 1_0, \\ 1_0 \circ (1_0 \odot 1_0), & (1_0 \circ 1_0) \odot 1_0, & 1_0 \odot (1_0 \odot 1_0), & (1_0 \odot 1_0) \odot 1_0 \end{array}$$

Exercise 5.2. List all the arrows in Q_2 with $Q = \{a : 0 \to 1\}$.

Exercise 5.3. List all the arrows in Q_1 when $Q = \{s : 2 \to 2\}$.

Exercise 5.4. List all the arrows in Q_1 when Q consists of the two arrows $c : 0 \to 2$ and $d : 2 \to 0$.

We will call an arrow in a quiver Q_i a *formal combination of arrows in Q*. We let Q^* denote the quiver of all formal combinations of arrows in Q. In other words, $Q^* = \bigcup_i Q_i$. For example, the following is a formal combination of arrows in $Q = \{a : 0 \to 1\}$:

$$((1_1 \circ a) \odot (1_4 \odot 1_0)) \circ ((a \odot a) \odot 1_3) \circ ((a \odot 1_1) \odot a) \tag{5.4}$$

Exercise 5.5. Let b denote the arrow in (5.4). Find $m, n \in \mathbb{Z}_{\geq 0}$ such that $b : m \to n$. Find the smallest i such that $b \in Q_i$.

Exercise 5.6. Explain why the expression $(c \odot c) \circ (c \odot (1_2 \odot d))$ is not a valid formal combination of the arrows $c : 0 \to 2$ and $d : 2 \to 0$.

An equivalence relation on arrows

We now introduce an equivalence relation on formal combinations of arrows which will allow us to simplify expressions. See Appendix A for the definition of an equivalence relation. Given a quiver Q, we let $\sim$ denote the weakest equivalence relation on Q^* such that the following hold for all $l, l', m, m', n, n' \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} &\text{if } a \sim b \text{ and } c \sim d \text{ then } a \circ c \sim b \circ d \\ &\qquad\qquad \text{for all } a, b : m \to n \text{ and } c, d : l \to m, \end{aligned} \tag{5.5}$$

$$\text{if } a \sim a' \text{ and } b \sim b' \text{ then } a \odot b \sim a' \odot b' \text{ for all arrows } a, a', b, b', \tag{5.6}$$

$$a \circ 1_m \sim a \sim 1_n \circ a \text{ for all } a : m \to n, \tag{5.7}$$

$$(a \circ b) \circ c \sim a \circ (b \circ c) \text{ for all } a : m \to n, b : l \to m, c : k \to l, \tag{5.8}$$

$$1_n \odot 1_{n'} \sim 1_{n+n'}, \tag{5.9}$$

$$\begin{aligned} &(a \odot a') \circ (b \odot b') \sim (a \circ b) \odot (a' \circ b') \\ &\qquad\qquad \text{for all } a : m \to n, a' : m' \to n', b : l \to m, b' : l' \to m', \end{aligned} \tag{5.10}$$

$$a \odot 1_0 \sim a \sim 1_0 \odot a \text{ for all arrows } a, \tag{5.11}$$

$$(a \odot b) \odot c \sim a \odot (b \odot c) \text{ for all arrows } a, b, c. \tag{5.12}$$

The quotient $Q^*/\sim$ is another quiver. Given an arrow $a : m \to n$ Q^*, we will abuse notation in the standard way by also writing $a : m \to n$ for the corresponding equivalence class in $Q^*/\sim$. Note, however, that $a = b$ in $Q^*/\sim$ if and only if $a \sim b$ in Q^*.

For example, if $Q = \{a : 0 \to 1\}$ then the arrows in Q_1 which are distinct when viewed as arrows in $Q^*/\sim$ are

$$a : 0 \to 1, \quad a \odot a : 0 \to 2, \quad 1_n : n \to n \quad (n \in \mathbb{Z}_{\geq 0}),$$

$$a \odot 1_n : n \to n+1 \quad (n \in \mathbb{Z}_{>0}), \quad 1_n \odot a : n \to n+1 \quad (n \in \mathbb{Z}_{>0}).$$

Exercise 5.7. Suppose $Q = \{a : 0 \to 1\}$. Which of the arrows in Q_2 are distinct when viewed as arrows in $Q^*/{\sim}$?

Exercise 5.8. Suppose $Q = \{s : 2 \to 2\}$ as in Exercise 5.3. Which of the arrows in Q_1 are distinct when viewed as arrows in $Q^*/{\sim}$?

Exercise 5.9. Suppose $Q = \{c : 0 \to 2,\, d : 2 \to 0\}$ as in Exercise 5.4. Which of the arrows in Q_1 are distinct when viewed as arrows in $Q^*/{\sim}$?

Free monoidal categories generated by one object

The equivalences (5.5)–(5.12) are precisely the requirements necessary for the formal operations $\circ$ and $\odot$ on arrows in a quiver to prescribe a strict monoidal category. More precisely, we have the following:

Definition 5.10. Given a quiver Q consisting of arrows between nonnegative integers, the *free monoidal category generated by a single object and the arrows in* Q is the strict monoidal category $(\mathcal{F}(Q), \odot, 1_0)$ where:

- $\mathcal{F}(Q)$ is the category with $\operatorname{Ob}\mathcal{F}(Q) = \mathbb{Z}_{\geq 0}$ and morphisms

  $$\operatorname{Hom}_{\mathcal{F}(Q)}(m, n) = \{\text{arrows in } Q^*/{\sim} \text{ of the form } m \to n\}.$$

  The composition map in $\mathcal{F}(Q)$ is given by

  $$\begin{array}{rcl} \operatorname{Hom}_{\mathcal{F}(Q)}(m,n) \times \operatorname{Hom}_{\mathcal{F}(Q)}(l,m) & \to & \operatorname{Hom}_{\mathcal{F}(Q)}(l,n) \\ (a,b) & \mapsto & a \circ b \end{array} \tag{5.13}$$

- $\odot$ is a bifunctor on $\mathcal{F}(Q)$ defined on objects by $n \odot n' = n + n'$ and on morphisms by

  $$\begin{array}{rcl} \operatorname{Hom}_{\mathcal{F}(Q)}(m,n) \times \operatorname{Hom}_{\mathcal{F}(Q)}(m',n') & \to & \operatorname{Hom}_{\mathcal{F}(Q)}(m+m', n+n') \\ (a,b) & \mapsto & a \odot b \end{array} \tag{5.14}$$

Note that (5.5) and (5.6) imply that the maps (5.13) and (5.14) are well-defined. It follows from (5.7) and (5.8) that $\mathcal{F}(Q)$ satisfies (C2) and (C3) respectively. The reader is encouraged to verify that $\mathcal{F}(Q)$ also satisfies (C1). Hence $\mathcal{F}(Q)$ is indeed a category. To see that $\odot$ defines a bifunctor on $\mathcal{F}(Q)$, note that (B1) and (B2) follow from (5.9) and (5.10) respectively. Finally, (M1) and (M2) follow from (5.11) and (5.12), whence $(\mathcal{F}(Q), \odot, 1_0)$ is indeed a strict monoidal category.

Exercise 5.11. Suppose $x : 0 \to 0$ and $y : 0 \to 0$ are arrows in a quiver Q. Use (5.7), (5.10), and (5.11) to prove that the following hold in $\mathcal{F}(Q)$:

$$x \odot y = x \circ y = y \odot x = y \circ x.$$

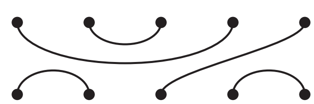
Exercise 5.12. Let Q denote a quiver. Prove that every non-identity morphism in $\mathcal{F}(Q)$ can be written as a $\circ$-product of finitely many morphisms of the form $1_m \odot a \odot 1_n$ with $a \in Q$. Hints: Start with an arbitrary formal combination of arrows in Q and induct on the number of arrows from Q in the expression. Use the property $a \odot b = (a \odot 1_n) \circ (1_m \odot b)$ which follows from (5.7) and (5.10).

The term *free* in Definition 5.10 alludes to the fact that there are no relationships among arrows in $\mathcal{F}(Q)$ other than the relationships that hold in every strict monoidal category. As a consequence, we have the following:

Proposition 5.13. ***(Universal property of free monoidal categories)*** *Suppose Q is a quiver of arrows $a_\lambda : m_\lambda \to n_\lambda$ where each $m_\lambda, n_\lambda \in \mathbb{Z}_{\geq 0}$ and λ ranges over some indexing set Λ. Given a strict monoidal category $(\mathcal{C}, \odot, \mathbb{1})$, an object $x \in \operatorname{Ob}\mathcal{C}$, and a morphism $f_\lambda : x^{\odot m_\lambda} \to x^{\odot n_\lambda}$ in $\mathcal{C}$ for each $\lambda \in \Lambda$; there exists a unique strict monoidal functor $G : \mathcal{F}(Q) \to \mathcal{C}$ such that $G(1) = x$ and $G(a_\lambda) = f_\lambda$ for each $\lambda \in \Lambda$.*

Proof. I should probably write this proof. For now, let's call it an exercise. □

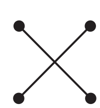
Example 5.14. Let Q be the quiver with two arrows $c : 0 \to 2$ and $d : 2 \to 0$ and write $\mathcal{F} = \mathcal{F}(Q)$. Let C and D denote the unique Temperley-Lieb diagrams of types $0 \to 2$ and $2 \to 0$ respectfully. Let $G : \mathcal{F} \to \mathcal{TL}$ be the unique strict monoidal functor with $G(1) = 1$, $G(c) = C$, and $G(d) = D$, as prescribed by the universal property of $\mathcal{F}$. Then G maps $(1_1 \odot c \odot 1_2) \circ (d \odot c \odot 1_1 \odot d)$ to the following diagram:



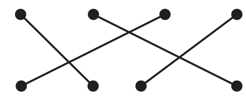
Since every Temperley-Lieb diagram can be built by stacking C's, D's, and identity diagrams (see §5.1), it follows that the functor G is surjective on morphisms. However, G is not injective on morphisms (see Exercise 5.15), hence G is not an isomorphism.

Exercise 5.15. Let $\mathcal{F}$ and G be as in Example 5.14. Show there exists a functor $H : \mathcal{F} \to \mathcal{P}$ such that $H((d \odot 1_1) \circ (1_1 \odot c)) \neq H(1_1)$. Why does that inequality imply that $(d \odot 1_1) \circ (1_1 \odot c) \neq 1_1$ in $\mathcal{F}$? Now, show that $G((d \odot 1_1) \circ (1_1 \odot c)) = G(1_1)$ and conclude that G is not injective on morphisms.

Exercise 5.16. Set $Q = \{s : 2 \to 2\}$ and let $G : \mathcal{F}(Q) \to \mathcal{S}$ be the unique strict monoidal functor with $G(1) = 1$ and

$G(s) =$  .

Find a morphism a in $\mathcal{F}(Q)$ such that

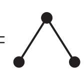
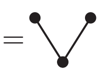
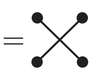
$G(a) =$  .

Is G surjective and/or injective on morphisms? Are there any other strict monoidal functors $H : \mathcal{F}(Q) \to \mathcal{S}$ with $H(1) = 1$?

Exercise 5.17. Let Q be the quiver with five arrows

$$a : 0 \to 1, \quad b : 1 \to 0, \quad m : 2 \to 1, \quad w : 1 \to 2, \quad \text{and} \quad s : 2 \to 2.$$

Let $G : \mathcal{F}(Q) \to \mathcal{P}$ be the unique strict monoidal functor with $G(1) = 1$ and

$G(a) =$ $G(b) =$ $G(m) =$  $G(w) =$  $G(s) =$  .

Is G surjective on morphisms? Is G injective on morphisms?

Exercise 5.18. With Exercise 5.17 in mind, find a quiver Q with four arrows and a strict monoidal functor $G : \mathcal{F}(Q) \to \mathcal{NC}$ that is surjective on morphisms.

5.3 Quotients of categories by relations

Congruence relations and quotient categories

Given a category $\mathcal{C}$, a *congruence relation on* $\mathcal{C}$ is an equivalence relation $\approx$ on the set of all morphisms in $\mathcal{C}$ such that

$$f \circ g \approx f' \circ g' \text{ whenever } f \approx f' \text{ and } g \approx g'. \tag{5.15}$$

The corresponding *quotient category* $\mathcal{C}/{\approx}$ is defined by setting $\mathrm{Ob}(\mathcal{C}/{\approx}) = \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}/\approx}(m, n) = \mathrm{Hom}_{\mathcal{C}}(m, n)/{\approx}$ for all $m, n \in \mathrm{Ob}\,\mathcal{C}/{\approx}$. It follows from (5.15) that the composition map in $\mathcal{C}$ induces a well-defined composition in $\mathcal{C}/{\approx}$. A *monoidal congruence relation* on a strict monoidal category $(\mathcal{C}, \odot, \mathbb{1})$ is a congruence relation $\approx$ on $\mathcal{C}$ such that

$$f \odot g \approx f' \odot g' \text{ whenever } f \approx f' \text{ and } g \approx g'. \tag{5.16}$$

In this case, the quotient category $\mathcal{C}/\approx$ inherits the structure of a strict monoidal category from $\mathcal{C}$.

Exercise 5.19. The goal of this exercise is to prove the claims above concerning quotient categories. Assume $\approx$ is a congruence relation on a category $\mathcal{C}$. For each morphism f in $\mathcal{C}$, write $[f] = \{g \ : \ g \approx f\}$ for the equivalence class containing f. Prove the following:

1. The composition map on $\mathcal{C}/\!\approx$ defined by setting $[f] \circ [g] = [f \circ g]$ is well-defined.

2. The definition of $\mathcal{C}/\!\approx$ satisfies (C1), (C2), and (C3).

3. If $\approx$ is monoidal, then setting $[f] \odot [g] = [f \odot g]$ prescribes a well-defined bifunctor on $\mathcal{C}/\!\approx$. Moreover, $(\mathcal{C}/\!\approx, \odot, \mathbb{1})$ satisfies (M1) and (M2).

Exercise 5.20. Given permutation diagrams B and D, write $B \approx D$ if and only if $(-1)^B = (-1)^D$ (see (3.1)). Prove that $\approx$ is a monoidal congruence relation on $(\mathcal{S}, \otimes, 0)$. Describe the quotient category $\mathcal{S}/\!\approx$.

Given a congruence relation $\approx$ on a category $\mathcal{C}$, there is an associated *quotient functor* $\Pi : \mathcal{C} \to \mathcal{C}/\!\approx$ which is the identity map on objects and maps each morphism to its equivalence class. If $\approx$ is a monoidal congruence relation, then Π is a strict monoidal functor. The following proposition is the most important property of quotient categories, and will be useful in subsequent sections.

Proposition 5.21. ***(Universal property of quotient categories)*** *Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories, $\approx$ is a congruence relation on $\mathcal{C}$, and $F : \mathcal{C} \to \mathcal{D}$ is a functor. If $F(f) = F(g)$ in $\mathcal{D}$ whenever $f \approx g$ in $\mathcal{C}$, then there exists a unique functor $\widetilde{F} : \mathcal{C}/\!\approx \to \mathcal{D}$ such that $F = \widetilde{F} \circ \Pi$. Moreover, if F is a strict monoidal functor and $\approx$ is a monoidal congruence relation, then $\widetilde{F}$ is also a strict monoidal functor.*

Proof. Define $\widetilde{F}$ on objects by setting $\widetilde{F}(n) = F(n)$. To defined $\widetilde{F}$ on morphisms, note that each morphism in $\mathcal{C}/\!\approx$ is of the form $[f] = \{g \ : \ g \approx f\}$ for some f in $\mathcal{C}$. Set $\widetilde{F}([f]) = F(f)$. To see that $\widetilde{F}$ is well-defined on morphisms, assume $[f] = [g]$ in $\mathcal{C}/\!\approx$. Then $f \approx g$ in $\mathcal{C}$, whence $F(f) = F(g)$ in $\mathcal{D}$, as desired. Next, we have $\widetilde{F} \circ \Pi(f) = \widetilde{F}([f]) = F(f)$, from which it follows that $F = \widetilde{F} \circ \Pi$. The remainder of the proof is left as an exercise. □

Exercise 5.22. Complete the proof of Proposition 5.21 by showing that $\widetilde{F}$ is strict monoidal functor whenever F is strict monoidal functor and $\approx$ is a monoidal congruence relation.

Generators and relations

The most important examples of quotient categories for our purposes are quotients of the free monoidal categories $\mathcal{F}(Q)$ for various quivers Q. Such quotients can be viewed as imposing additional relations on formal combinations of arrows in Q that do not already follow from (5.5)–(5.12). To be more precise, let R denote a set of relations among morphisms in $\mathcal{F}(Q)$, and let $\approx$ denote the weakest congruence class on $\mathcal{F}(Q)$ such that the relations in R hold. We will call the quotient category $\mathcal{F}(Q)/\!\approx$ the *free monoidal category generated by a single object and morphisms Q subject to the relations R.*

As a running example for the rest of this section, let $\mathcal{E}$ denote the free monoidal category generated by a single object and a morphism $\varepsilon : 0 \to 0$ subject to the relations

$$\varepsilon^2 \approx 1_0, \quad \varepsilon \odot 1_n \approx 1_n \odot \varepsilon \quad (\text{for all } n \in \mathbb{Z}_{\geq 0}). \tag{5.17}$$

Note that by Exercise 5.11 you can interpret ε^2 as $\varepsilon \circ \varepsilon$ or $\varepsilon \odot \varepsilon$. We will describe $\mathcal{E}$ in detail by the end of this section. To start, notice that since the only generating morphism for $\mathcal{E}$ is an endomorphism, we have $\mathrm{Hom}_{\mathcal{E}}(m,n) = \varnothing$ unless $m = n$.

Given a morphism $a \in \mathrm{Hom}_{\mathcal{F}(Q)}(m,n)$, it is customary to abuse notation and also write a for the corresponding equivalence class in $\mathrm{Hom}_{\mathcal{F}(Q)/\approx}(m,n)$. However, it is important to remember that $a = b$ in $\mathcal{F}(Q)/\approx$ if and only if $a \approx b$ in $\mathcal{F}(Q)$. Breaking this down even further, $a = b$ in $\mathcal{F}(Q)/\approx$ exactly when b can be obtained form a by a sequence of the relations among (5.5)–(5.12) and the defining relations for $\approx$. For example, we have the following in $\mathcal{E}$:

$$(1_5 \odot \varepsilon) \odot 1_6 = (\varepsilon \odot 1_5) \odot 1_6 = \varepsilon \odot (1_5 \odot 1_6) = \varepsilon \odot 1_{11}.$$

The equality on the left follows from one of the defining relations for $\mathcal{E}$ which gives $1_5 \odot \varepsilon \approx \varepsilon \odot 1_5$ in $\mathcal{F}(Q)$. The other two equalities follow from (5.12) and (5.9) respectively.

Exercise 5.23. For each $n \in \mathbb{Z}_{\geq 0}$, write $\varepsilon_n = \varepsilon \odot 1_n$. Show that every morphism in $\mathcal{E}$ is equal to either ε_n or 1_n for some $n \in \mathbb{Z}_{\geq 0}$.

Showing equality of two morphisms in category defined by generators and relations often amounts to some computation with the defining relations. On the other hand, it can be a bit trickier to show that two morphisms in such a category are not equal. For example, you might suspect that $\varepsilon_n \neq 1_n$ in $\mathcal{E}$; but how do you prove it? In this case you could argue that two expressions cannot be equal unless they have an equal parity of ε's appearing (the defining relations can only add/remove ε's in pairs). However, there is a different approach involving functors which is preferable, especially when working with more complicated systems of generators and relations. The method is to find some functor $\mathcal{E} \to \mathcal{D}$ such that the images of ε_n and 1_n are not equal in $\mathcal{D}$. To cook up such a functor, we often use the universal properties of free monoidal categories (Proposition 5.13) and quotient categories (Proposition 5.21).

For example, let $\mathcal{F}$ denote the free monoidal category generated by a single object and the arrow $\varepsilon : 0 \to 0$ and consider the strict monoidal category $(\mathbf{Mat}, \otimes, 1)$. By Proposition 5.13 there exists a unique strict monoidal functor

$$G : \mathcal{F} \to \mathbf{Mat}$$

with $G(1) = 1$ and $G(\varepsilon) = -I_1$.* To see that G induces a functor from $\mathcal{E}$ to $\mathbf{Mat}$, we need to check that the defining relations for $\mathcal{E}$ are preserved by G:

*Note that G maps all objects to 1. Indeed, $G(n) = 1^{\otimes n} = 1$ for all $n \in \mathbb{Z}_{\geq 0}$. Hence, G maps $\mathrm{End}_{\mathcal{E}}(n)$ to $\mathrm{End}_{\mathbf{Mat}}(1) = \{\lambda I_1 \ : \ \lambda \in \mathbb{C}\}$. In other words, each morphism in $\mathcal{F}$ is assigned a complex number by G.

Exercise 5.24. Show that $G(\varepsilon^2) = G(1_0)$ and $G(\varepsilon \odot 1_n) = G(1_n \odot \varepsilon)$ for all $n \in \mathbb{Z}_{\geq 0}$. Explain why if follows that $G(a) = G(b)$ whenever $a = b$ in $\mathcal{E}$.

It follows from the previous exercise and Proposition 5.21 that there exists a strict monoidal functor $\widetilde{G} : \mathcal{E} \to \mathbf{Mat}$ with $\widetilde{G}(1) = 1$ and $\widetilde{G}(\varepsilon) = -I_1$.

Exercise 5.25. Show that $\widetilde{G}(\varepsilon_n) \neq \widetilde{G}(1_n)$ for all $n \in \mathbb{Z}_{\geq 0}$. Explain why it follows that $\varepsilon_n \neq 1_n$ in $\mathcal{E}$.

At this point we know everything about the category $\mathcal{E}$. Indeed, we knew from the start that the only morphisms in $\mathcal{E}$ were endomorphisms. It follows from Exercises 5.23 and 5.25 that each object n admits exactly two endomorphisms: 1_n and ε_n. Moreover, operations on these endomorphisms are completely controlled by the following multiplication table:

	1	ε
1	1	ε
ε	ε	1

The multiplication table above can be view as a multiplication table for either operation $\circ$ or $\odot$ with any compatible subscripts on the 1's and ε's. For instance, $\varepsilon_3 \odot \varepsilon_4 = 1_7$, $1_4 \odot \varepsilon_2 = \varepsilon_6$, and $\varepsilon_4 \circ \varepsilon_4 = 1_4$.

To close this section, notice that after the initial definition of the congruence relation $\approx$ to define $\mathcal{E}$, we replace "$\approx$" with "$=$" in all computations with the understanding that "$=$" in $\mathcal{E}$ amounts to "$\approx$" in $\mathcal{F}(Q)$. In the upcoming section we will go one step further by removing the notation "$\approx$" even in the definition of a category given by generators and relations. For example, we allow ourselves to *define* $\mathcal{E}$ as the the free monoidal category generated by a single object and a morphism $\varepsilon : 0 \to 0$ subject to the relations

$$\varepsilon^2 = 1_0, \quad \varepsilon \odot 1_n = 1_n \odot \varepsilon \quad (\text{for all } n \in \mathbb{Z}_{\geq 0}).$$

Compare the slight modification in notation from (5.17).

5.4 A Presentation of $\mathcal{TL}$

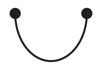
The goal of this section is to prove the following theorem:

Theorem 5.26. *$\mathcal{TL}$ is isomorphic to the free monoidal category generated by a single object and two morphisms $c : 0 \to 2$, $d : 2 \to 0$ subject to the relations:*

$$(d \odot 1_1) \circ (1_1 \odot c) = 1_1 = (1_1 \odot d) \circ (c \odot 1_1), \quad d \circ c = 1_0. \tag{5.18}$$

For the remainder of the section, let $\mathcal{FTL}$ denote the category given by generators and relations as described in the previous theorem.

Exercise 5.27. Use Example 5.14 and Proposition 5.21 to show that there exists a strict monoidal functor $F : \mathcal{FTL} \to \mathcal{TL}$ with

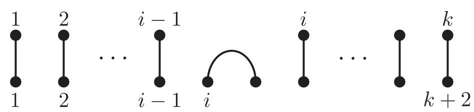
$F(c) =$  $F(d) =$ 

Show that F is bijective on objects.

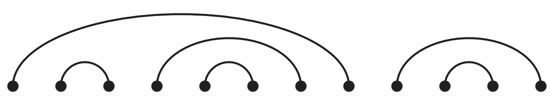
To prove Theorem 5.26 we must show that the functor F described above is bijective on morphisms. It's not hard to show subjectivity, but injectivity requires some care. Towards that end, we first describe a unique way to decompose certain Temperley-Lieb diagrams.

A normal form for Temperley-Lieb diagrams

Write $D_i^{(k)} = 1_{i-1} \otimes F(d) \otimes 1_{k-i+1}$. In other words, $D_i^{(k)}$ is the following Temperley-Lieb diagram of type $k+2 \to k$.

$D_i^{(k)} =$ 

We can decompose any Temperley-Lieb diagram of type $2n \to 0$ as a $\bullet$-product of $D_i^{(k)}$'s. For example, if

$D =$ 

then $D = D_1^{(0)} \bullet D_2^{(2)} \bullet D_4^{(4)} \bullet D_5^{(6)} \bullet D_9^{(8)} \bullet D_{10}^{(10)}$. I general, such a decomposition is not unique. For example, the diagram above can also be decomposed as $D = D_1^{(0)} \bullet D_2^{(2)} \bullet D_1^{(4)} \bullet D_2^{(6)} \bullet D_4^{(8)} \bullet D_5^{(10)}$. However, it turns out that we do get a unique decomposition if we require the subscripts on the $D_i^{(k)}$'s to be strictly increasing (as in the former decomposition).

Proposition 5.28. *For each Temperley-Lieb diagram $D : 2n \to 0$ there exists a unique sequence of integers $1 = i_1 < i_2 < \cdots < i_n$ with each $i_k < 2k$ such that*

$$D = D_{i_1}^{(0)} \bullet D_{i_2}^{(2)} \bullet \cdots \bullet D_{i_n}^{(2n-2)}.$$

Exercise 5.29. Prove Proposition 5.28. Be sure to prove the uniqueness part of the proposition; it will be crucial later.

Calculations in $\mathcal{FTL}$

Let $c_i^{(k)} : k \to k+2$ and $d_i^{(k)} : k+2 \to k$ denote the following morphisms in the category $\mathcal{FTL}$:

$$c_i^{(k)} = 1_{i-1} \odot c \odot 1_{k-i+1}, \quad d_i^{(k)} = 1_{i-1} \odot d \odot 1_{k-i+1}.$$

In particular, note that $F(d_i^{(k)}) = D_i^{(k)}$ for all $1 \le i \le k+1$. Also, note that by Exercise 5.12, every morphism in $\mathcal{FTL}$ is either an identity morphism or a $\circ$-product of finitely many $c_i^{(k)}$'s and $d_i^{(k)}$'s. As a consequence, we have

$$\mathrm{Hom}_{\mathcal{FTL}}(m,n) = \varnothing \text{ unless } m \text{ and } n \text{ have the same parity.} \tag{5.19}$$

The following exercises will allow us to prove Theorem 5.26.

Exercise 5.30. Prove that the following hold in $\mathcal{FTL}$:

$$\begin{aligned}
d_i^{(k)} \circ c_i^{(k)} &= 1_k && (1 \le i \le k+1)\\
d_i^{(k)} \circ c_{i+1}^{(k)} &= 1_k && (1 \le i < k+1)\\
d_i^{(k)} \circ c_{i-1}^{(k)} &= 1_k && (1 < i \le k+1)\\
d_i^{(k)} \circ c_j^{(k)} &= c_{j-2}^{(k-2)} \circ d_i^{(k-2)} && (1 < i+1 < j \le k+1)\\
d_i^{(k)} \circ c_j^{(k)} &= c_j^{(k-2)} \circ d_{i-2}^{(k-2)} && (1 < j+1 < i \le k+1)\\
d_i^{(k)} \circ d_j^{(k+2)} &= d_j^{(k)} \circ d_{i+2}^{(k+2)} && (1 \le j \le i \le k+1)\\
c_i^{(k)} \circ c_j^{(k-2)} &= c_{j+2}^{(k)} \circ c_i^{(k-2)} && (1 \le i \le j \le k-1)
\end{aligned}$$

Exercise 5.31. Use the previous exercise to prove that every morphism in $\mathrm{Hom}_{\mathcal{FTL}}(2n,0)$ can be written in the form

$$d_{i_1}^{(0)} \circ d_{i_2}^{(2)} \circ \cdots \circ d_{i_n}^{(2n-2)}$$

with $1 = i_1 < i_2 < \cdots < i_n$.

Exercise 5.32. Find morphisms $\gamma_n : 0 \to 2n$ and $\delta_n : 2n \to 0$ in $\mathcal{FTL}$ such that $F(\gamma_n) = \Cup_n$ and $F(\delta_n) = \Cap_n$ (see Exercise 1.19). Show that

$$(\delta_n \odot 1_n) \circ (1_n \odot \gamma_n) = 1_n \quad \text{and} \quad (1_n \odot \delta_n) \circ (\gamma_n \odot 1_n) = 1_n$$

in $\mathcal{FTL}$. Hint: Write γ_n in terms of $c_i^{(k)}$'s and δ_n in terms of $d_i^{(k)}$'s. Then induct on n using Exercise 5.30.

Exercise 5.33. (Compare with Exercise 1.19) Use the previous exercise to prove the map $\mathrm{Hom}_{\mathcal{FTL}}(m,n) \to \mathrm{Hom}_{\mathcal{FTL}}(m+n,0)$ given by $x \mapsto \delta_n \circ (x \odot 1_n)$ is a bijection with inverse $y \mapsto (y \odot 1_n) \circ (1_m \odot \gamma_n)$.

Proof of Theorem 5.26

Let $m_1, m_2 \in \mathbb{Z}_{\ge 0}$. To prove Theorem 5.26 we must show that map

$$\mathrm{Hom}_{\mathcal{FTL}}(m_1,m_2) \to \mathrm{Hom}_{\mathcal{TL}}(m_1,m_2)$$

given by $x \mapsto F(x)$ is a bijection. Both Hom-sets are empty unless m_1 and m_2 have the same parity (see (5.19)), so we may assume $m_1 + m_2 = 2n$ for some $n \in \mathbb{Z}_{\geq 0}$. Consider the following square of maps where ϕ and ψ are the bijections prescribed by Exercises 5.33 and 1.19 respectively:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{FTL}}(m_1, m_2) \arrow[r, "F"] \arrow[d, "\phi"'] & \mathrm{Hom}_{\mathcal{TL}}(m_1, m_2) \arrow[d, "\psi"'] \\
\mathrm{Hom}_{\mathcal{FTL}}(2n, 0) \arrow[r, "F"] & \mathrm{Hom}_{\mathcal{TL}}(2n, 0).
\end{tikzcd}$$

The following computation shows that the square commutes:

$$\begin{aligned}
F \circ \phi(x) &= F(\delta_n \circ (x \odot 1_n)) && \text{(Definition of } \phi\text{)} \\
&= F(\delta_n) \bullet (F(x) \otimes 1_n) && \text{(}F \text{ is a strict monoidal functor)} \\
&= \Cap_n \bullet (F(x) \otimes 1_n) && \text{(Definition of } \delta_n\text{)} \\
&= \psi \circ F(x) && \text{(Definition of } \psi\text{)}
\end{aligned}$$

Thus we have $F \circ \phi = \psi \circ F$, or equivalently $F = \psi^{-1} \circ F \circ \phi$. Therefore, since both ψ and ϕ are bijections, it suffices to prove F is bijective on morphisms of the form $2n \to 0$.

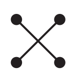
Exercise 5.34. Use Proposition 5.28 and Exercise 5.31 to prove that the map

$$\mathrm{Hom}_{\mathcal{FTL}}(2n, 0) \to \mathrm{Hom}_{\mathcal{TL}}(2n, 0)$$

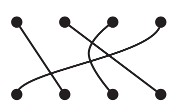
given by $x \mapsto F(x)$ is a bijection for each $n \in \mathbb{Z}_{\geq 0}$. This completes the proof of Theorem 5.26. □

5.5 A Presentation of $\mathcal{S}$

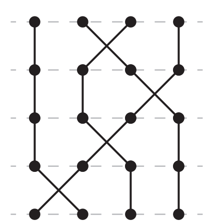
The standard generator for the category $\mathcal{S}$ is the cross:



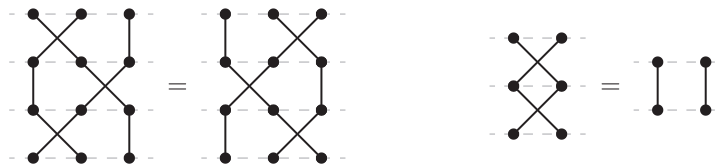
Just as we can decompose any Temperley-Lieb diagram into cups and caps, we can decompose any permutation diagram into crosses. For example, the permutation diagram



admits the following decomposition:



The fundamental relations for crosses are the following:



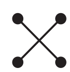
The relation above on the left is called the *braid relation*. The goal of this section is to prove that $\mathcal{S}$ is generated by the cross subject to the two relations given above. More precisely, we will prove the following:

Theorem 5.35. *$\mathcal{S}$ is isomorphic to the free monoidal category generated by a single object and one morphism $s : 2 \to 2$ subject to the relations:*

$$(s \odot 1_1) \circ (1_1 \odot s) \circ (s \odot 1_1) = (1_1 \odot s) \circ (s \odot 1_1) \circ (1_1 \odot s), \quad s \circ s = 1_2. \tag{5.20}$$

Let $\mathcal{FS}$ denote the category given by generators and relations as described in the previous theorem.

Exercise 5.36. Use Exercise 5.16 and Proposition 5.21 to show that there exists a strict monoidal functor $F : \mathcal{FS} \to \mathcal{S}$ with

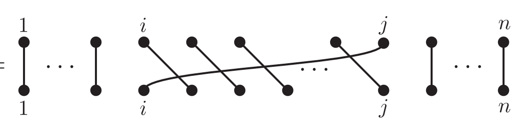
$F(s) =$ 

Show that F is bijective on objects.

As in the Temperley-Lieb case, to prove Theorem 5.35 we must show that F is bijective on morphisms.

A normal form for permutation diagrams

Given integers $1 \le i \le j \le n$ we define the permutation diagram $X_{i,j}^{(n)} : n \to n$ as follows. First set $X_{i,i}^{(n)} = 1_n$ for all i. Now, for $i < j$ we set

$X_{i,j}^{(n)} =$ 

The $X_{i,j}^{(n)}$'s can be used to uniquely decompose permutation diagrams:

Proposition 5.37. *For each permutation diagram $D : n \to n$ there exists a unique sequence of integers $j_1, j_2, \ldots, j_{n-1}$ with $i \le j_i \le n$ for each i, such that*

$$D = X_{1,j_1}^{(n)} \bullet X_{2,j_2}^{(n)} \bullet \cdots \bullet X_{n-1,j_{n-1}}^{(n)}.$$

Exercise 5.38. Prove Proposition 5.37. Hint: Show that D can be written in the form $D = X_{1,j_1}^{(n)} \bullet (1_1 \otimes D')$ for some permutation diagram $D' : n-1 \to n-1$. Then induct on n.

Calculations in $\mathcal{FS}$

Let $s_i^{(n)} : n \to n$ denote the following morphism in the category $\mathcal{FS}$:

$$s_i^{(n)} = 1_{i-1} \odot s \odot 1_{n-i-1}.$$

Note that by Exercise 5.12, every morphism in $\mathcal{FS}$ is either an identity morphism or a $\circ$-product of finitely many $s_i^{(n)}$'s. As a consequence, we have

$$\mathrm{Hom}_{\mathcal{FS}}(m,n) = \varnothing \text{ unless } m = n. \tag{5.21}$$

Now, given integers $1 \le i \le j \le n$ we set

$$x_{i,j}^{(n)} = \begin{cases} 1_n, & \text{if } i = j; \\ s_j \circ s_{j-1} \circ \cdots s_{i+1} \circ s_i & \text{if } i < j \le n. \end{cases}$$

In particular, note that $F(x_{i,j}^{(n)}) = X_{i,j}^{(n)}$ for all $1 \le i \le j \le n$.

The following analogues to Exercises 5.30 and 5.31 will allow us to prove Theorem 5.35:

Exercise 5.39. Prove that the following hold in $\mathcal{FS}$:

$$\begin{aligned} s_i^{(n)} \circ s_i^{(n)} &= 1_n & & (1 \le i \le n-1) \\ s_i^{(n)} \circ s_{i+1}^{(n)} \circ s_i^{(n)} &= s_{i+1}^{(n)} \circ s_i^{(n)} \circ s_{i+1}^{(n)} & & (1 \le i \le n-2) \\ s_i^{(n)} \circ s_j^{(n)} &= s_j^{(n)} \circ s_i^{(n)} & & (|i-j| > 1) \end{aligned}$$

Exercise 5.40. Use the previous exercise to prove that every morphism in $\mathrm{End}_{\mathcal{FS}}(n)$ can be written in the form

$$x_{1,j_1}^{(n)} \circ x_{2,j_2}^{(n)} \circ \cdots \circ x_{n-1,j_{n-1}}^{(n)}$$

for some integers $j_1, j_2, \ldots, j_{n-1}$ with $i \le j_i \le n$ for each i.

Proof of Theorem 5.35

To prove Theorem 5.35 we must show that map $\mathrm{Hom}_{\mathcal{FS}}(m,n) \to \mathrm{Hom}_{\mathcal{S}}(m,n)$ given by $\sigma \mapsto F(\sigma)$ is a bijection for all $m, n \in \mathbb{Z}_{\ge 0}$. Both Hom-sets are empty unless $m = n$ (see (5.21)), hence it suffices to show that map

$$\mathrm{End}_{\mathcal{FS}}(n) \to \mathrm{End}_{\mathcal{S}}(n) \tag{5.22}$$

is a bijection. By Exercise 5.37, every morphism in $\mathrm{End}_{\mathcal{S}}(n)$ is of the form

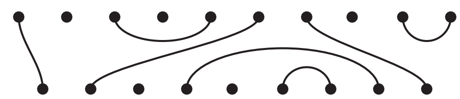
$$X_{1,j_1}^{(n)} \bullet \cdots \bullet X_{n-1,j_{n-1}}^{(n)} = F(x_{1,j_1}^{(n)} \circ \cdots \circ x_{n-1,j_{n-1}}^{(n)}).$$

It follows that (5.22) is surjective.

Exercise 5.41. Use Proposition 5.37 and Proposition 5.40 to prove that (5.22) is injective. This completes the proof of Theorem 5.35. $\square$

5.6 Problems: presentations of other diagram categories

Problem 5.42. (The Motzkin category) A partition diagram is called *Motzkin* if it is non-crossing and each part consists of either one or two elements. For instance, every Temperley-Lieb diagram is Motzkin. Here's an example of a Motzkin diagram which is not Temperley-Lieb:



It's easy to check that Motzkin diagrams are closed under the operations $\bullet$ and $\otimes$. Hence, the collection of all Motzkin diagrams form a subcategory $\mathcal{M}$ of $\mathcal{P}$, and $(\mathcal{M}, \otimes, 0)$ is a strict monoidal category. Prove the $\mathcal{M}$ is isomorphic to the free monoidal category generated by a single object and three morphisms

$$b : 1 \to 0, \quad c : 0 \to 2, \quad d : 2 \to 0,$$

subject to relations (5.18) along with

$$(b \odot 1_1) \circ c = (1_1 \odot b) \circ c, \quad b \circ (b \odot 1_1) \circ c = 1_0.$$

Problem 5.43. (Presentation of $\mathcal{B}$) Prove the $\mathcal{B}$ is isomorphic to the free monoidal category generated by a single object and three morphisms

$$c : 0 \to 2, \quad d : 2 \to 0, \quad s : 2 \to 2,$$

subject to relations (5.18), (5.20), and the following:

$$d \circ s = d, \quad (d \odot 1_1) \circ (1_1 \odot s) = (1_1 \odot d) \circ (s \odot 1_1). \tag{5.23}$$

Problem 5.44. (Presentations of $\mathcal{NC}$ and $\mathcal{TL}^{ev}$) Prove the $\mathcal{NC}$ and $\mathcal{TL}^{ev}$ are both isomorphic to the free monoidal category generated by a single object and four morphisms

$$a : 0 \to 1, \quad b : 1 \to 0, \quad m : 2 \to 1, \quad w : 1 \to 2,$$

subject to the relations:

$$\begin{aligned} m \circ (m \odot 1_1) &= m \circ (1_1 \odot m), & m \circ (1_1 \odot a) &= 1_1 = m \circ (a \odot 1_1), \\ (w \odot 1_1) \circ w &= (1_1 \odot w) \circ w, & (1_1 \odot b) \circ w &= 1_1 = (b \odot 1_1) \circ w, \\ b \circ a &= 1_0, & m \circ w &= 1_1, \\ (m \odot 1_1) \circ (1_1 \odot w) &= w \circ m, & (1_1 \odot m) \circ (w \odot 1_1) &= w \circ m. \end{aligned} \tag{5.24}$$

Problem 5.45. (Presentation of $\mathcal{P}$) Prove the $\mathcal{P}$ is isomorphic to the free monoidal category generated by a single object and five morphisms

$$a : 0 \to 1, \quad b : 1 \to 0, \quad m : 2 \to 1, \quad w : 1 \to 2, \quad s : 2 \to 2,$$

subject to relations (5.20), (5.24), and the following:

$$\begin{aligned} s \circ (1_1 \odot a) &= a \odot 1_1, & (1_1 \odot m) \circ (s \odot 1_1) \circ (1_1 \odot s) &= s \circ (m \odot 1_1), \\ (1_1 \odot b) \circ s &= b \odot 1_1, & (1_1 \odot s) \circ (s \odot 1_1) \circ (1_1 \odot w) &= (w \odot 1_1) \circ s. \end{aligned}$$

Appendix A

Equivalence relations

A *relation R on a set X* is an endomorphism $R \in \mathrm{End}_{\mathbf{Rel}}(X)$. Recall that this means $R \subseteq X \times X$. A relation R on a set X is called *reflexive* if $(x, x) \in R$ for all $x \in X$. We call R *symmetric* if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$. We call R *transitive* if $(x, y), (y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$. A reflexive, symmetric, transitive relation is called an *equivalence relation.*

Exercise A.1. Find 8 different relations $R_1, \ldots, R_8$ on the set $\{1, 2, 3\}$ which agree with the following truth table:

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
reflexive	T	T	T	T	F	F	F	F
symmetric	T	T	F	F	T	T	F	F
transitive	T	F	T	F	T	F	T	F