$\sqrt[n]{2}$-Binary Look and Say Sequences

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1 Introduction

The look-and-say sequence was first introduced by John Conway. Let's look at a standard look and say sequence:

$$11 \to 21 \to 1211 \to 111221 \to 312211 \to 13112211 \to \cdots \tag{1.1}$$

The sequence above starts with the *seed* 11. It has two 1's. To create a subsequent term in the sequence, we simply "say what we see". As a result, the next term in the sequence is 21. 21 has one 2 followed by one 1. Thus, the next term is 1211. As we continue this pattern, we get one 1, one 2 and two 1's. As we continue, we see that the terms in the sequence above appear to be growing. Conway estimated the ratio of the lengths of terms to approach.

$$1.303577269\ldots$$

We now refer to this number as *Conway's constant*, which implies that each subsequent term in the sequence (1.1) is approximately 30% longer than the previous term. Conway's constant is the largest real root of the following irreducible polynomial:

$$\begin{aligned}
&\lambda^{71} - \lambda^{69} - 2\lambda^{68} - \lambda^{67} + 2\lambda^{66} + 2\lambda^{65} + \lambda^{64} - \lambda^{63} - \lambda^{62} - \lambda^{61} - \lambda^{60} - \lambda^{59} + 2\lambda^{58}\\
&+ 5\lambda^{57} + 3\lambda^{56} - 2\lambda^{55} - 10\lambda^{54} - 3\lambda^{53} - 2\lambda^{52} + 6\lambda^{51} + 6\lambda^{50} + \lambda^{49} + 9\lambda^{48} - 3\lambda^{47}\\
&- 7\lambda^{46} - 8\lambda^{45} - 8\lambda^{44} + 10\lambda^{43} + 6\lambda^{42} + 8\lambda^{41} - 5\lambda^{40} - 12\lambda^{39} + 7\lambda^{38} - 7\lambda^{37} + 7\lambda^{36}\\
&+ \lambda^{35} - 3\lambda^{34} + 10\lambda^{33} + \lambda^{32} - 6\lambda^{31} - 2\lambda^{30} - 10\lambda^{29} - 3\lambda^{28} + 2\lambda^{27} + 9\lambda^{26} - 3\lambda^{25}\\
&+ 14\lambda^{24} - 8\lambda^{23} - 7\lambda^{21} + 9\lambda^{20} + 3\lambda^{19} - 4\lambda^{18} - 10\lambda^{17} - 7\lambda^{16} + 12\lambda^{15} + 7\lambda^{14} + 2\lambda^{13}\\
&- 12\lambda^{12} - 4\lambda^{11} - 2\lambda^{10} + 5\lambda^{9} + \lambda^{7} - 7\lambda^{6} + 7\lambda^{5} - 4\lambda^{4} + 12\lambda^{3} - 6\lambda^{2} + 3\lambda - 6
\end{aligned}$$

This sequence is governed by the linear transformation of 92-dimensional vector space, which further illustrates the complication mathematics behind the "say what you see" sequence. After the introduction of Conway's standard look and say sequence several variations of look and say sequence have been introduced. (see e.g. [EBGSN+2, EBGSN+1], [OM], [Mor], [SS]).

In this paper we will explore a different non-standard look and say sequence, which we call the $\sqrt[n]{2}$-binary look and say sequence. In this paper we fix the number system to $\sqrt[n]{2}$ binary base.

In section 2 we introduce the $\sqrt[n]{2}$-binary number system and show how to represent numbers in $\sqrt[n]{2}$-binary.

In section 3 we will the we will look into the structure of look and say sequence with seed 0for some cases of $\sqrt[n]{2}$-Binary. More superficially when binary $n = 1^j0^k, 10^k$ and $n = 1^20^k$.

Likewise in section 4, We found out the lemma to find frequent elements for $\sqrt[n]{2}$-Binary look and say sequence with all possible seeds in a case basis for different cases of n. In the section we also looked at cases when $n = 1, 2$ and 3.

2 $\sqrt[n]{2}$-Binary Number System

We know, the binary number consists of two digits; 1 and 0. In our primary and secondary school, our understanding of binary numbers in confined to base 2. However, there are different binary number systems. Grey code is one of such examples.

Here, we will look at the family of one more binary number system. We call it base $\sqrt[n]{2}$-binary number system. For convenience we will refer base 2 binary numbers as binary and we write $(m)_2$ for binary representation of m. Similarly, we will write $[m]_n$ for $\sqrt[n]{2}$-binary representation of m where $m \in \mathbb{R}$.

The decimal to $\sqrt[n]{2}$-binary conversion is similar to decimal to binary conversion. Rather than 2 we have $\sqrt[n]{2}$ as our base. For example, when $n = 2$

$$12 = 1(\sqrt{2})^6 + 0(\sqrt{2})^5 + 1(\sqrt{2})^4 + 0(\sqrt{2})^3 + 0(\sqrt{2})^2 + 0(\sqrt{2})^1 + 0(\sqrt{2})^0$$

$$[12]_2 = 1010000$$

let's look at some more conversions:

Decimal	Binary	$\sqrt{2}$-Binary	$\sqrt[3]{2}$-Binary	$\sqrt[4]{2}$-Binary
0	0	0	0	0
1	1	1	1	1
2	10	100	1000	10000
3	11	101	1001	10001
4	100	10000	1000000	100000000
5	101	10001	1000001	100000001

From the conversions above we can see that $\sqrt[n]{2}$-binary representations are obtained by inserting $n - 1$ 0s between each bit in binary representation. This enables us to represent any decimal numbers in $\sqrt[n]{2}$-binary number system using binary number system.

2.1 Writing $\sqrt[n]{2}$-binary in exponent form.

From the section above, we see that $\sqrt[n]{2}$-binary representation of a number can have many 0s in it. So, for our convenience we will write $\sqrt[n]{2}$-binary representations in exponent form. The exponent will determine the number of consecutive 0's or 1's in the $\sqrt[n]{2}$-binary/binary representation. For example $1^3 0^2$ would represent 3 consecutive 1's followed by 2 consecutive 0's i.e 11100.

2.2 Writing 3, $n - 1$ and $2n - 1$ in $\sqrt[n]{2}$-binary form

From section 2, we can represent any number in $\sqrt[n]{2}$ binary from its binary form. For example let's look at number 3.

$$\begin{aligned} (3)_2 &= 11 \\ [3]_n &= 10^{n-1}1 \end{aligned} \tag{2.1}$$

Further let's look at how we can write $n-1$ and $2n-1$ in binary system. To subtract 1 from a number n_2 (say 11000000), we flip all the bits after the rightmost 1 bit, (we get 11111111). Finally, we flip the rightmost 1 bit also, (we get 10111111) to get the answer. So in our case when

$$\begin{aligned} (n)_2 &= 1^j 0^k \\ (n-1)_2 &= 1^{j-1} 0 1^k \end{aligned} \tag{2.2}$$

Now, $(2n)_2$ is just multiplying the binary bit by 2 i.e increasing the power of each 2 by 1. Therefore, When

$$\begin{aligned} (n)_2 &= 1^j 0^k \\ (2n)_2 &= 1^j 0^{k+1} \\ (2n-1)_2 &= 1^{j-1} 0 1^{k+1} \end{aligned} \tag{2.3}$$

Then, in $\sqrt[n]{2}$-binary we get

$$\begin{aligned} [n-1]_n &= (10^{n-1})^{j-2} 10^{n-1} 00^{n-1} (10^{n-1})^{k-1} 1 \\ &= (10^{n-1})^{j-2} 10^{2n-1} (10^{n-1})^{k-1} 1 \\ [2n-1]_n &= (10^{n-1})^{j-2} 10^{n-1} 00^{n-1} (10^{n-1})^{k} 1 \\ &= (10^{n-1})^{j-2} 10^{2n-1} (10^{n-1})^{k} 1 \end{aligned} \tag{2.4}$$

From this general form we can convert any $(n)_2$ to $[n]_n$ form for every $j \geq 2$. For example when $j = 2$

$$\begin{aligned} (n)_2 &= 1^2 0^k \\ [n-1]_n &= 10^{2n-1} (10^{n-1})^{k-1} 1 \\ [2n-1]_n &= 10^{2n-1} (10^{n-1})^{k} 1 \end{aligned} \tag{2.5}$$

When $j = 1$,

$$\begin{aligned} (n)_2 &= 10^k \\ (n-1)_2 &= 1^k \\ (2n)_2 &= 10^{k+1} \\ (2n-1)_2 &= 1^{k+1} \\ [n-1]_n &= (10^{n-1})^{k-1} 1 \\ [2n-1]_n &= (10^{n-1})^{k} 1 \end{aligned} \tag{2.6}$$

3 $\sqrt[n]{2}$-Binary Look and Say Sequence with Seed 0

Using $\sqrt[n]{2}$-binary representations gives us a new way to *say* what we see when generating a look and say sequences. For example, if we look at 1111 then we see four 1's. Since the $\sqrt[n]{2}$-binary representation of four is 10^{2n-1} we would say $10^{2n-1}1$. As in the standard case, repeatedly applying this say-what-you-see operation will generate a look and say sequence.

Consider the look and say sequence starting with the seed 0 for $n = 2$ First, we see one 0 so the next term is 10. From 10 we see one 1 and one 0 so the next term is 1110. Now, looking at 1110 we see three 1's followed by one 0; since $[3]_2$ is 101, the next term will be 101110. Continuing on in this manner gives us the following look and say sequence:

$$0 \to 10 \to 1110 \to 101110 \to 1110101110 \to 1011101110101110 \to \cdots \tag{3.1}$$

In the following sections we will look into the structure of look and say sequences for some $\sqrt[n]{2}$-binary cases, where we fix n.

3.1 Case 1: when $(n)_2 = 10^k$

In this section we want to look at binary look and say sequence with base $\sqrt[n]{2}$ where we fix $n = 2^k$.

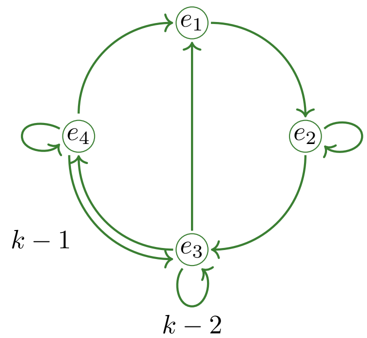
Theorem 3.1. *When $(n)_2 = 1^2 0^k$, the frequent elements that appear in look and say sequence with seed 0 are $e_1 = 10, e_2 = 1^3 0, e_3 = 10^{n-1}, e_4 = 1^3 0^{n-1}$, where $e_1 \to e_2$, $e_2 \to e_3 e_2$, $e_3 \to e_4 e_3^{k-2} e_1$, $e_4 \to e_3 e_4 e_3^{k-2} e_1$.*

Proof. The first 2 terms of $\sqrt[n]{2}$-binary look and say sequence starting with seed 0 are $e_1 = 10$ and $e_2 = 1110$ regardless of the number system used.

$$0 \to 10 \to 1110 \to \cdots$$

Using section 2, we have the following decay:
$e_2 = 1110 \to 10^{n-1}1110 = e_3 e_2$,
$e_3 = 10^{n-1} \to 11(10^{n-1})^{k-1}10 = 1110^{n-1}(10^{n-1})^{k-2}10 = e_4 e_3^{k-2} e_1$,
$e_4 = 1^3 0^{n-1} \to 10^{n-1}11(10^{n-1})^{k-1}10 = 10^{n-1}1110^{n-1}(10^{n-1})^{k-2}10 = e_3 e_4 e_3^{k-2} e_1$. □

3.1.1 Decay Graph



(3.2)

In the graph above each arrow from e_j to e_i indicates an occurrence of e_i in the decay of e_j. For example, since e_4 decays into $e_3^{k-1}e_4e_1$, we have $k-1$ arrows from e_4 to e_3; one arrow from e_4 to itself and one arrow from e_4 to e_1. In other words, the number of arrows from e_j to e_i in the decay graph is equal to the i, j-entry in the decay matrix. (3.1.2)

Now, we see e_4 decays into e_3 $k-1$ times and e_3 decays into itself $k-2$ times. Therefore, eventually the ratio of the elements in a term will be approximately $0:0:1:0$.

3.1.2 Characteristic polynomials and growth rates

To estimate the growth rate of the sequence, we can encode the frequent elements into a 4×4 decay matrix. With k frequent elements, the decay matrix is the $k \times k$ matrix whose i, j-entry is the number of times e_i occurs in the decay of e_j. In other words, the j^th column is the compound vector corresponding to the decay of e_j. It follows from theorem 3.1.

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & k-2 & k-1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of the 4×4 matrix is

$$-\lambda^2(-\lambda^2 + k\lambda + 2 - k)$$

The corresponding maximal real eigenvalue

$$\lambda = \frac{k + \sqrt{k^2 - 4k + 8}}{2}$$

The ratio of the length of the terms of the Look and Say sequence approaches the maximal real eigenvalue.

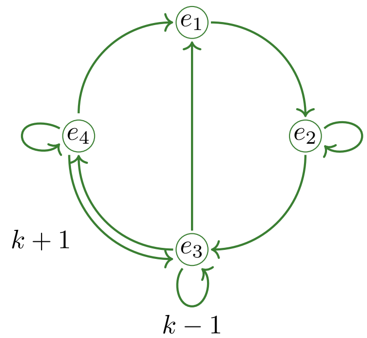
3.2 Case 2 : when $(n)_2 = 1^2 0^k$

3.2.1 Decomposition of frequent elements

Theorem 3.2. *When $(n)_2 = 1^2 0^k$, the frequent elements that appear in look and say sequence with seed 0 are $e_1 = 10$, $e_2 = 1^3 0$, $e_3 = 10^{n-1}$, and $e_4 = 1^3 0^{2n-1}$, where $e_1 \to e_2, e_2 \to e_3 e_2, e_3 \to e_4 e_3^{k-1} e_1$, and $e_4 \to e_3^{k+1} e_4 e_1$.*

Proof. Since we are starting with seed 0,
$0 \to 10 = e_1$
$e_1 = 10 \to 1110 = 1^3 0 = e_2$
$e_2 = 1^3 0 \to 10^{n-1}1110 = 10^{n-1}1^3 0 = e_3 e_2$
$e_3 = 10^{n-1} \to 1110^{2n-1}(10^{n-1})^{k-1}10 = 1^3 0^{2n-1}(10^{n-1})^{k-1}10 = e_4 e_3^{k-1} e_1$
$e_4 = 1^3 0^{2n-1} \to 10^{n-1}1110^{2n-1}(10^{n-1})^k 10 = 10^{n-1}1^3 0^{2n-1}(10^{n-1})^k 10 = e_3^{k+1} e_4 e_1$
□

3.2.2 Decay Graph



(3.3)

Looking at the decay graph we see e_4 decays into e_3 $k+1$ times and e_3 decays into itself $k-1$ times. Therefore, eventually the ratio of the elements in a term will be approximately $0:0:1:0$.

3.2.3 Decay Matrix

Form Section 3.1.2, we know how to create a decay matrix. So, The decay matrix in this case is:

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & k-1 & k+1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Using technology, the characteristic polynomial of this matrix is

$$\lambda^4 - \lambda^3 - 2\lambda^2 - k\lambda^3 + k\lambda^2 + \lambda.$$

The ratio of length of the terms of the look and say sequence approach the maximal real root of the characteristic polynomial of the decay matrix. Using technology we get the following maximal real root for the given characteristic polynomial:

$$\lambda = \frac{\alpha}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}(-k^2+k-7)}{3\alpha} + \frac{k+1}{3}$$

when,

$$\alpha = \sqrt[3]{(2k^3 - 3k^2 + 3\sqrt{3}\sqrt{-k^4+2k^3-15k^2+14k-49} + 15k - 7}$$

3.3 Case 3: When $(\mathbf{n})_2 = 1^j0^k$

3.3.1 Decomposition of frequent elements

Theorem 3.3. *When $(n)_2 = 1^j0^k$ the frequent elements that appear in look and say sequence with seed* 0 *are* $e_1 = 10$ *,* $e_2 = 1^30$*,* $e_3 = 10^{n-1}$*,* $e_4 = 10^{2n-1}$ *and*

$e_5 = 1^3 0^{n-1}$, *where* $e_1 \to e_2, e_2 \to e_3 e_2, e_3 \to e_5 e_3^{j+k-4} e_4 e_1$, $e_4 \to e_5 e_3^{j+k-3} e_4 e_1$ *and* $e_5 \to e_3^{j+k-3} e_5 e_4 e_1$.

Proof. Since we are starting with seed 0,

$$
\begin{aligned}
0 &\to 10 = e_1 \\
e_1 &= 10 \to 1110 = 1^3 0 = e_2 \\
e_2 &= 1^3 0 \to 10^{n-1} 1110 = 10^{n-1} 1^3 0 = e_3 e_2 \\
e_3 &= 10^{n-1} \to 11(10^{n-1})^{j-2} 10^{2n-1} (10^{n-1})^{k-1} 10 = 1^3 0^{n-1} (10^{n-1})^{j-3} 10^{2n-1} (10^{n-1})^{k-1} 10 \\
&= e_5 e_3^{j+k-4} e_4 e_1 \\
e_4 &= 10^{2n-1} \to 11(10^{n-1})^{j-3} 10^{2n-1} (10^{n-1})^{k} 10 = 1^3 0^{n-1} 10^{2n-1} (10^{n-1})^{k} 10 \\
&= e_5 e_3^{j+k-3} e_4 e_1 \\
e_5 &= 1^3 0^{n-1} \to 10^{n-1} 11 (10^{n-1})^{j-2} 10^{2n-1} (10^{n-1})^{k-1} 10 = 10^{n-1} 1^3 0^{n-1} (10^{n-1})^{j-3} 10^{2n-1} (10^{n-1})^{k-1} 10 \\
&= e_3^{j+k-3} e_5 e_4 e_1.
\end{aligned}
$$

□

3.3.2 Decay Matrix

Form Section 3.1.2, we know how to create a decay matrix. The decay matrix in this case is

$$
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & j+k-4 & j+k-3 & j+k-3 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}
$$

Using technology, we get the following characteristic polynomial for the given matrix:

$$j\lambda^4 - j\lambda^3 - \lambda^5 - \lambda^4 + 4\lambda^3 - \lambda^2 + k\lambda^4 - k\lambda^3$$

The ratio of length of the terms in this case will approach the maximal real root of the given characteristic polynomial.

4 $\sqrt[n]{2}$-Binary Look and Say Sequence with all possible seeds

In this section we establish a general pattern for all look-and-say sequences where n is not fixed in $\sqrt[n]{2}$-binary base. Given any seed, the Cosmological Lemma allows us to establish persistent elements in all look-and-say sequences.

4.1 Results of Day One and Day Two

We refer to the terms as n-days-old if they appear after n^{th} decay in the sequence. If we take 1^y0^z as a seed, it is zero days old and it's decay appears in form $[y]_n1[z]_n0$, which is now one day old.

Lemma 4.1. *(Day One Lemma) No more than 3 consecutive 1's will appear in the $\sqrt[n]{2}$-binary look and say sequence starting from day one.*

Proof. Consider seed $1^y0^z \to [y]_n1[z]_n0$. Suppose $(y)_2 = a_j \dots a_1a_0$ and $(z)_2 = b_k \dots b_1b_0$, then

$$[y]_n1[z]_n0 = a_j0^{n-1} \dots a_20^{n-1}a_10^{n-1}a_01b_k0^{n-1} \dots b_20^{n-1}b_10^{n-1}b_00$$

Expressing y and z in $\sqrt[n]{2}$-binary number system does not include any consecutive 1's. Depending on the value of y, a_0 can either be 1 or 0. On the other hand, b_k is digit 1. Therefore, the decay of 1^y0^z can contain at most 3 consecutive 1's:

$$a_01b_k.$$

□

Since $y \in \{1, 2, 3\}$, we only need to consider the following representations of y.

y	$[y]_n$
1	1
2	10^n
3	$10^{n-1}1$

Lemma 4.2. *(Day Two Lemma) The number of consecutive 0's in a two day-old string are limited to $1, n$, and $in \pm 1$ for $i \in \mathbb{N}$.*

Proof. Consider a day old element $1^y0^z \to [y]_n1[z]_n0$. Suppose $(z)_2 = b_k \dots b_1b_0$. By Day One Lemma we know that $y \in \{1, 2, 3\}$. When $y = 1$,

$$[y]_n1[z]_n0 = 11b_k0^{n-1} \dots b_20^{n-1}b_10^{n-1}b_00.$$

When $y = 2$,

$$[y]_n1[z]_n0 = 10^n1b_k0^{n-1} \dots b_20^{n-1}b_10^{n-1}b_00.$$

When $y = 3$,

$$[y]_n1[z]_n0 = 10^{n-1}11b_k0^{n-1} \dots b_20^{n-1}b_10^{n-1}b_00.$$

Suppose $b_0 = 1$ in $[y]_n1[z]_n0$. Then we can present the string in form

$$[y]_n1b_k0^{n-1} \dots b_20^{n-1}b_10^{n-1}10.$$

As a result, $[z]_n 0$ will run 0 and $in-1$ number of 0's for some $i < k$, depending on b values. Meanwhile $[y]_n$ runs either n or $n-1$ number of 0's. Suppose $b_0 = 0$. Then the string will be in form

$$[y]_n 1 b_k 0^{n-1} \dots b_2 0^{n-1} b_1 0^{n+1}.$$

Therefore, $[z]_n 0$ can run $in-1$ and $in+1$ number of 0's for some $i < k$, depending on b values. As a result, starting day two, we limit the runs of 0's to $1, n$ and $in \pm 1$ for $i \in \mathbb{N}$. □

4.2 Cosmological Theorems

In this section, we consider cases where $n > 1$ for $\sqrt[n]{2}$-binary look and say sequence.

Lemma 4.3. *(Cosmological lemma) The decay of any one day element $1^y 0^z$ will have less than z consecutive 0's whenever $z^n < 2^{z-1}$ and $z > n$.*

Proof. $1^y 0^z \to [y]_n 1 [z]_n 0$. By Lemma 4.1 we know that the maximal run of 0's in $[y]_n$ is n.We assume that $z > n$ in this proposition. Therefore, it suffices to check where length in $[z]_n 0$ is less than $[z]_n$.

Let l denote the length of the maximal run of 0's in $[z]_n 0$. We will assume that $z^n < 2^{z-1}$ and show that $l < z$. If

$$(z)_2 = b_k \dots b_2 b_1 b_0$$

then

$$[z]_n = b_k 0^{n-1} \dots b_2 0^{n-1} b_1 0^{n-1} b_0.$$

Since there can be maximum nk digits of 0 in $[z]_n$, the maximum number of 0's in $[z]_n 0$ is $nk + 1$ and $l \le nk + 1$.

$$z^n < 2^{z-1}$$

$$\log_2 z^n < z - 1$$

$$\log_2 z^n + 1 < z$$

The smallest number you can write with $k+1$ bits is $n = 2^k$ and the largest number with $k+1$ bits is $n = 2^{k+1} - 1$. Therefore, $2^k \le z \le 2^{k+1} - 1$. We are concerned with the smallest number with $k+1$ bits, since it is the smallest number that has largest consecutive 0's. Since $2^k \le z \le 2^{k+1} - 1$, then $k \le \log_2 z$. Therefore

$$nk \le n \log_2 z$$

$$nk \le \log_2 z^n$$

$$l \le nk + 1 \le \log_2 z^n + 1 \le z$$

Therefore

$$l \le z.$$

□

The Cosmological Lemma allows us to determine finite number of seeds that one needs to check to find all persistent elements in any $\sqrt[n]{2}$-binary look and say sequence. The following Cosmological Theorems for $n = 3$ and $n = 4$ illustrate under what conditions the theorems will give us frequent elements for any value of n.

Theorem 4.4. *(Cosmological Theorem for n=2) Terms of any $\sqrt[3]{2}$-binary look and say sequence are eventually compounds of the following 7 elements:* $e_1 = 1$ $e_2 = 10, e_3 = 11, e_4 = 10^2, e_5 = 1^2 0, e_6 = 1^3 0, e_7 = 1^3 0^3$.

Proof. The $z^n < 2^{z-1}$ inequality holds true whenever $z \geq 6$. Along with Day One and Day Two Lemmas, this allows us to reduce the number of cases we need to check for when
$y \in \{1, 2, 3\}$
$z \in \{0, 1, 2, 3, 4, 5, 6\}$
In total, there are 21 seeds that require checking. Using Python, we generate the binary chemistry for 21 possible seeds, which returns the frequent elements listed in the theorem. □

Theorem 4.5. *(Cosmological Theorem for n=3) Terms of any $\sqrt[3]{2}$-binary look and say sequence are eventually compounds of the following 15 elements:* $e_1 = 1$ $e_2 = 10, e_3 = 11, e_4 = 10^2, e_5 = 1^2 0, e_6 = 10^3, e_7 = 1^2 0^2, e_8 = 1^3 0, e_9 = 1^3 0^2, e_{10} = 1^2 0^4, e_{11} = 1^2 0^5, e_{12} = 1^3 0^4, e_{13} = 1^3 0^5, e_{14} = 1^2 0^7, e_{15} = 1^3 0^7$.

Proof. The $z^n < 2^{z-1}$ inequality holds true whenever $z \geq 12$. Along with Day One and Day Two Lemmas, this allows us to reduce the number of cases we need to check for when
$y \in \{1, 2, 3\}$
$z \in \{0, 1, 2, 3, 4, 5, 7, 8, 10, 11\}$
In total, there are 30 seeds that require checking. Using Python, we generate the binary chemistry for 30 possible seeds, which returns the 15 frequent elements listed in the theorem. □

Theorem 4.6. *(Cosmological Theorem for n=4) Terms of any $\sqrt[4]{2}$-binary look and say sequence are eventually compounds of the following 12 elements:* $e_1 = 1$ $e_2 = 10, e_3 = 11, e_4 = 1^2 0, e_5 = 10^3, e_6 = 1^3 0, e_7 = 10^4, e_8 = 1^2 0^3, e_9 = 1^3 0^3, e_{10} = 1^3 0^6, e_{11} = 1^3 0^9, e_{12} = 1^3 0^1 1$.

Proof. The $z^n < 2^{z-1}$ inequality holds true whenever $z \geq 18$. Along with Day One and Day Two Lemmas, this allows us to reduce the number of cases we need to check for when
$y \in \{1, 2, 3\}$
$z \in \{0, 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17\}$
In total, there are 42 seeds that require checking. Using Python, we generate the binary chemistry for 42 possible seeds, which returns the 12 frequent elements listed in the theorem.

□

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