A BASIS THEOREM FOR THE JELLYFISH BRAUER CATEGORY $\mathcal{JB}(2)$

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1. Introduction
by Jonathan Comes

This paper is concerned with diagram categories related to the representation theory of the orthogonal and special orthogonal groups, $O(d)$ and $SO(d)$ respectively. The role of diagrams in the representation theory of these groups was initiated by Brauer in [Bra]. In that paper, Brauer used certain diagrams to construct algebras $B_m(d)$ equipped with surjective algebra maps

$$B_n(d) \to \mathrm{End}_{O(d)}(V^{\otimes n}) \tag{1.1}$$

where V is the natural d-dimensional representation of $O(d)$. These Brauer algebras are said to be in Schur-Weyl duality with the orthogonal groups, and they serve as a key tool in the study of representations of $O(d)$. Brauer also described a variation of his diagrams that form a basis for an $SO(d)$-analog of $B_n(d)$, but the multiplication rule for this algebra was not included. Instead in [Bra], Brauer writes "The rule for multiplication... can also be formulated. It is, however, more complicated and shall not be given here."

More recently, the relationship between Brauer diagrams and the representation theory of orthogonal groups[1] has been studied through the viewpoint of monoidal categories (see [LZ2]). In particular, one can use Brauer diagrams to construct the so-called Brauer category $\mathcal{B}(d)$ which admits a full monoidal functor

$$\mathcal{B}(d) \to \mathrm{Rep}(O(d)). \tag{1.2}$$

This viewpoint is an extension of Brauer's work in that the algebra maps (1.1) appear as the maps on endomorphism algebras induced by (1.2). Furthermore, in [LZ3] the authors explain how handle the $SO(d)$ case by adding an additional generator and a few new relations to the Brauer category, creating what they call the *enhanced Brauer category*. This additional generator has the appearance of a jellyfish, so following [Com] in this paper it will be called the *jellyfish Brauer category*, denoted $\mathcal{JB}(d)$. This category is equipped with a monoidal functor

$$F : \mathcal{JB}(d) \to \mathrm{Rep}(SO(d)). \tag{1.3}$$

The main result of [LZ3] is that F induces an equivalence of monoidal categories. In particular, F induces vector space isomorphisms on Hom-spaces.

Date: April 2023.

[1] In fact, the authors in [LZ1, LZ4] use Brauer diagrams to study the representation theory of more than just orthogonal groups. They study the representation theory of orthosymplectic supergroups.

This paper is concerned with the jellyfish Brauer category in the special case when $d = 2$. In this case, the functor (1.3) induces an isomorphism of vector spaces

$$JB_{2n}^0 \cong \mathrm{Hom}_{SO(2)}(V^{2n}, \mathbb{C}) \tag{1.4}$$

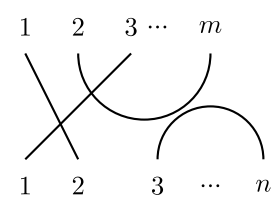
where JB_{2n}^0 denotes the appropriate space of linear combinations of jellyfish Brauer diagrams (see §3) and V is the natural 2-dimensional representation of $SO(2)$. It is a straightforward exercise in representation theory to show

$$\dim \mathrm{Hom}_{SO(2)}(V^{2n}, \mathbb{C}) = \binom{2n}{n}. \tag{1.5}$$

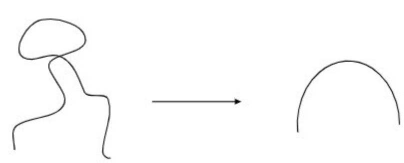
The expression $\binom{2n}{n}$ is closely related to a closed formula for the nth Catalan number (see §5.1). Moreover, the Catalan numbers count Temperley-Lieb diagrams, which are special types of Brauer diagrams (see Proposition 5.1). This suggests the existence of a diagram basis for JB_{2n}^0 consisting of certain jellyfish Temperley-Leib diagrams. The main result of this paper is such a diagram basis (see Corollary 7.5).

2. Definition of Brauer Diagrams

Given $m, n \in \mathbb{Z}_{\geq 0}$, let $[n] = \{j : 1 \leq j \leq n\}$, $[m]' = \{j' : 1 \leq j' \leq m\}$. A *Brauer diagram* of type $n \to m$ will look like:

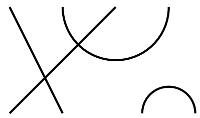
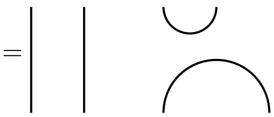


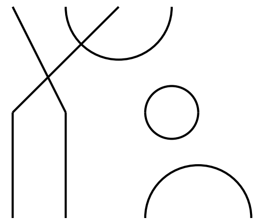
Which is a representation of a partition of $[n] \cup [m]'$ into pairs obtained by putting all n on the bottom and all m on the top, with strands connecting those nodes that are paired with each other. Note that this means any path taken between two endpoints is equal to any other path between those same endpoints. As a consequence, we see that we can "pull tight" any strand in a diagram. This means that any twists and deformities in a single strand can be smoothed out, so long as the strand's endpoints do not change. This can look something like this:



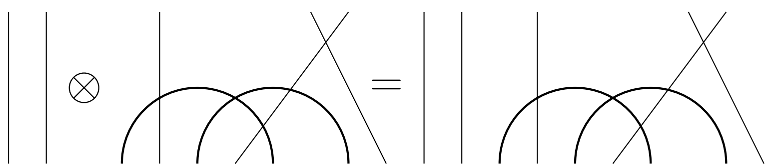
We can also use this method to locally deform Brauer diagrams in order to isolate something happening in that spot. We define B_n^m to be the space of complex linear combinations of Brauer diagrams of type $n \to m$. The Brauer algebra $B_n(d)$ mentioned in the introduction is the vector space B_n^n.

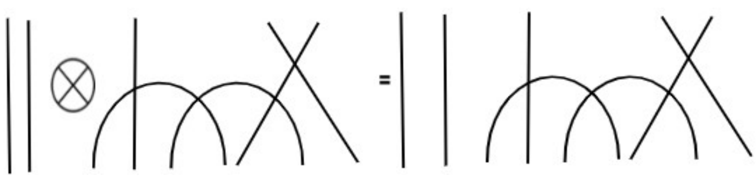
2.1. **Operations on Brauer Diagrams.** One property of Brauer diagrams is the capacity to both vertically and horizontally stack them. Stacking Brauer diagrams vertically gives composition in the Brauer category (and multiplication on the Brauer algebra). For example, if

$D_1 =$  and $D_2 =$ 

then $D_1 \circ D_2 =$ 

From this we see the possibility of loops arising. In the Brauer category $\mathcal{B}(d)$ a loop gives a factor of d. In this paper, $d = 2$, so that loops give a factor of 2. Thus $D_1 \circ D_2 = 2D_1$. Horizontal stacking gives the tensor product in the Brauer category, which might look like:

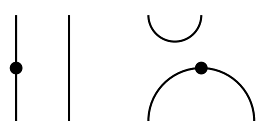




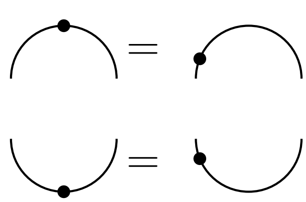
Now that we have established all the features of Brauer diagrams, we will add jellyfish to the picture.

3. Jellyfish and Jellyfish Brauer Diagrams

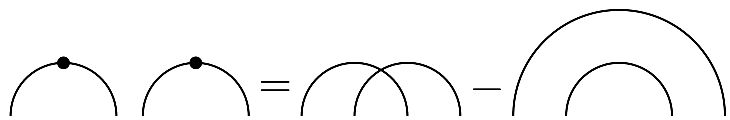
A jellyfish is a dot that can be added to any vertical strand on a Brauer diagram. Included is an example jellyfish Brauer diagram:

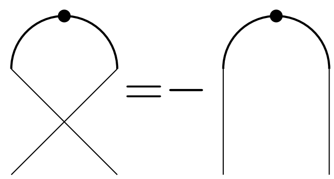


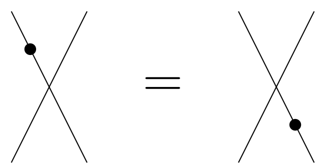
When a jellyfish is positioned on a horizontal strand section (on top of a cap or on the bottom of a cup), we define it to be equal to that jellyfish on the left side of the cap or cup.

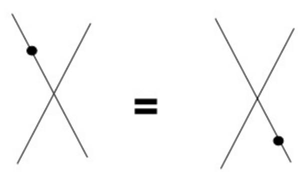


When discussing the space of diagrams with jellyfish included, we will use the notation JB_n^m. The jellyfish Brauer category JB(2) is defined by requiring the following three local relations:



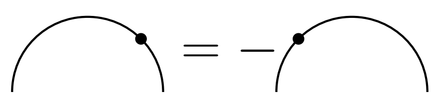




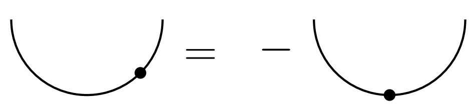


4. Relationships in JB diagrams

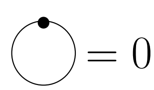
Theorem 4.1. *The following relations hold in JB diagrams*



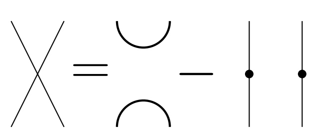
(4.1)



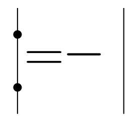
(4.2)



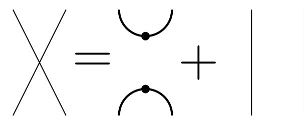
(4.3)



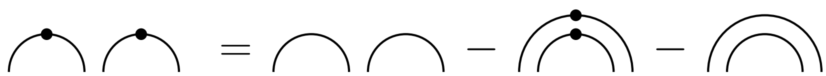
(4.4)



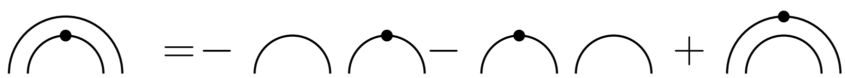
(4.5)



(4.6)

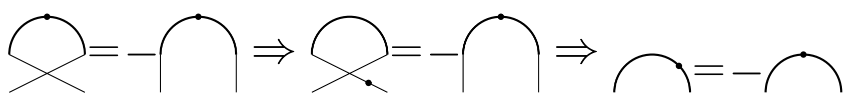


(4.7)



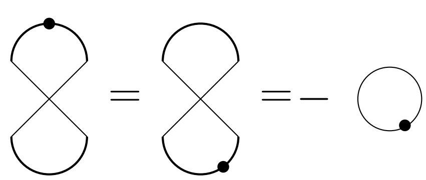
(4.8)

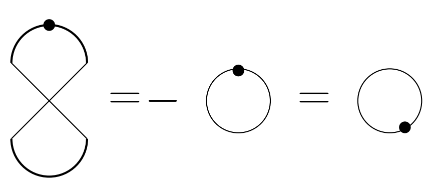
Proof. Relationship (4.1) is derived by using two of our definitions (dots passing through edge crossings, and crossed jellyfish being negated), and making an arbitrary decision. By moving our jellyfish leftward and down through the crossing, we have the following progression:



Thus we see that moving a jellyfish across a cap negates the diagram. This same logic also applies if we reflect our diagrams, giving us Relationship (4.2).

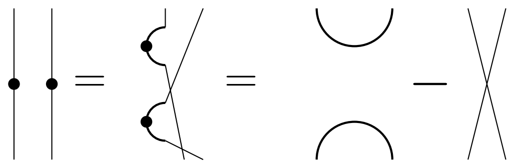
These relationships have immediate consequences for a jellyfish on a twisted loop, where we can choose to move through the edge crossing or undo the twisting first.





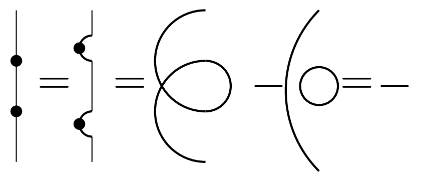
The above shows that a jellyfish on a closed loop will equal the same value as its negative, which is only true if it is equivalent to 0, which gives us (4.3).

We can define further relations by using our definition of two adjacent jellyfish on vertical connections instead of horizontally aligned connections. As we are only concerned about the connection of our edges, we have the equations below:

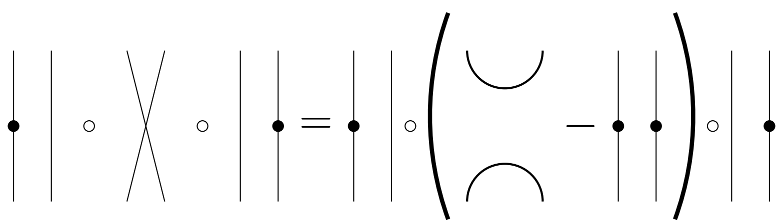


Thus be rearranging our terms, we have (4.4).

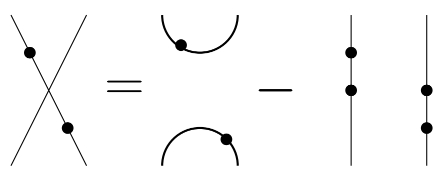
We can repeat the process, but on a singular line to instead have Relationship (4.5) below.



We can then use these relations and composition to make a second expansion of an edge crossing

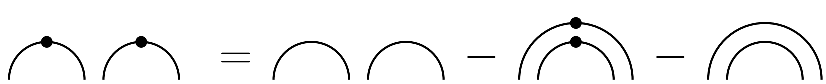


As our definition of composition is to stick diagrams together vertically, the effect of the diagrams above is to add a dot on the top left and bottom right of each diagram in Relationship (4.4):

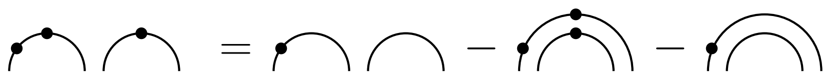


Upon this, we can use (4.5), (4.1), (4.2) and the definition of passing through edge crossings to have (4.6).

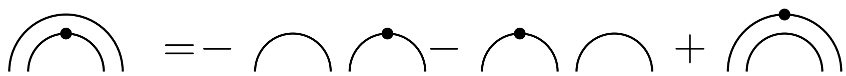
We can then apply these relationships locally. More specifically, as we are only concerned about the edges of our edge connections, we can 'stretch out' the connections, and thus consider a diagram to be equal to multiple diagrams composed together. Thus we can focus in on a local area of a diagram. For instance, applying (4.6) on our definition of adjacent jellyfish, then sliding dots as necessary, we have (4.7)



And upon (4.7), we can compose to add a dot on the first input (the leftmost edge) on each diagram to have the diagram below



Which can be simplified with (4.5) and reorganized into the desired relationship below, which is equivalent to (4.8)



□

These relationships can then be used to give further insight into JB diagrams. In particular (4.5) shows us that we only need consider diagrams where there are either 0 or 1 dot on each line. We will use this assumption for the rest of this paper without further reference.

5. Jellyfish-Temperly-Lieb

5.1. Catalan Numbers. The Catalan sequence is given by the following recursion relation:

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k} \tag{5.1}$$

where $n \geq 1$, and $C_0 = 1$. These numbers are equivalently given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{5.2}$$

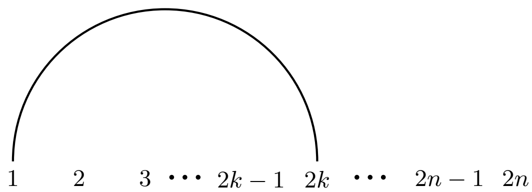
The Catalan sequence serves as a good starting point for counting the number of linearly independent jellyfish Brauer diagrams. We will first consider the undotted Temperley-Lieb (TL) diagrams, a subset of Brauer diagrams that contain no crossing strands.

Proposition 5.1. *The number of TL diagrams of type $2n \to 0$ is C_n.*

Proof. We will prove this by induction.

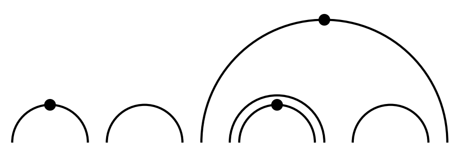
Base Case: Let $n = 0$. There is only one diagram with 0 strands, namely the empty diagram. Thus, the number of TL diagrams of type $0 \to 0$ is $1 = C_0$.

Induction Step: Assume that for any $m < n$, there are C_m possible TL diagrams of type $2m \to 0$. We will show that there are C_n possible diagrams of type $2n \to 0$ as well. First, we number the $2n$ ends of the n strands on the diagram from left to right. We will determine the second endpoint of the strand connected to endpoint 1. There are $2n - 1$ other endpoints. Notice, however, that to avoid any crossings, this strand must enclose an even number of endpoints. In other words, endpoint 1 can only connect to an even-numbered endpoint. This limits us to n ways to draw our first strand. So, we will arbitrarily draw the strand with endpoints 1 and $2k$, where $k \in \mathbb{N}$ such that $k \leq n$.



Now, the remaining endpoints are split into two sections. To avoid crossings, these sections cannot be connected by any strands and can therefore be treated as separate diagrams. Underneath the cap, there are $2k-2$ endpoints, and to the right of the cap, there are $2n-2k$ endpoints. Thus we have one TL diagram with type $2k-2 \to 0$, and one of type $2n-2k \to 0$, whose respective counts are C_{k-1} and C_{n-k} by the inductive hypothesis. So, for this choice of k we can draw $C_{k-1}C_{n-k}$ diagrams. But there are n choices of k, so we must now sum over each one. This gives a total count of $\sum_{k=1}^{n} C_{k-1}C_{n-k}$. But this is C_n by definition, so we see that for any n, the Catalan sequence describes the count of possible TL diagrams of type $2n \to 0$. □

5.2. **Jellyfish-Temperly-Lieb Span Jelly-Brauer.** Jellyfish Temperly Lieb diagrams (JTLs) are quite simply defined as JB diagrams without edge crossings. So an image like below is a JTL.

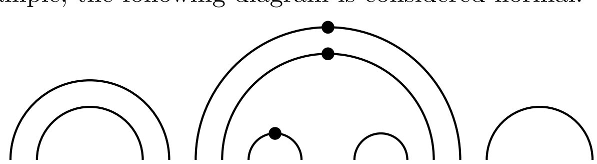


Theorem 5.2. *Each JB diagram can be expressed as a linear combination of JTL diagrams*

Proof. Given an arbitrary JB diagram, we can resolve any edge crossings by using (4.4) or (4.6) to instead have a linear combination of JTL diagrams. □

6. Normal Diagrams

Definition 6.1. A *normal* Jellyfish Temperly Lieb diagram is one where there exists no dots or there exists a strand such that only it and all strands above it are dotted. For example, the following diagram is considered normal.



Proposition 6.2. *A JTL diagram is normal if and only if neither an undotted strand with a dot below it nor two dotted strands next to each other occur in the diagram.*

(6.1)

Proof. If we have a normal JTL diagram, then by definition we do not have undotted strands above dotted strands, nor do we have any any other dotted strands next to each other. To show the converse, suppose we have a diagram that is not normal. Then there exists a strand that is dotted with at least one undotted strand above it or there exists two dots, neither of which is above the other. Thus if we have a dotted strand with an undotted strand above it, we have the first diagram. In the case where all dotted strands have a dotted strand above it, we either have at least two dotted strands on the top of the diagram or we have at least two dotted strands underneath another dotted strand, giving us two dots on the same level and thus the second diagram. □

Proposition 6.3. *The number of normal JTL diagrams of type $2n \to 0$ is $\binom{2n}{n}$.*

Proof. We know that for a diagram of n strands that we have C_n different un-dotted Temperly Lieb diagrams by Proposition 5.1. By Definition 6.1 we know that for any diagram with n strands we have one normal diagram per strand, plus one normal diagram for the undotted diagram. Thus as each undotted Temperly Lieb diagram has $n+1$ normal variations of it, we find that the total number of normal diagrams is then $(n+1)C_n = \frac{n+1}{n+1}\binom{2n}{n} = \binom{2n}{n}$. □

7. Normal Spanning

We now want to show that our previously defined normal diagrams span the set of JTL diagrams of type $2n \to 0$. There are a few results that will help show this.

7.1. **Eliminating Defect.**

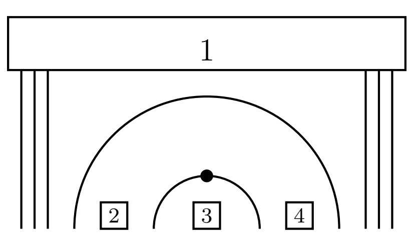
Definition 7.1. In a JTL diagram of type $2n \to 0$, we define the $defect$ of a dot to be the number of un-dotted strands that lie above it. The $defect$ of a diagram, then, is the sum of the defect of all dots in the diagram. We say a diagram is *without defect* when its defect is 0.

Lemma 7.2. *Any JTL diagram of type $2n \to 0$ can be written as a linear combination of diagrams without defect.*

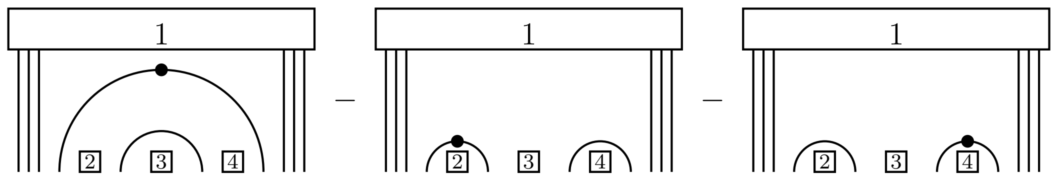
Proof. We will induct on the defect of the diagram.

Base case: The diagram has no defect and there is nothing to show.

Inductive step: Assume that any diagram with defect $d < n$ can be rewritten as a linear combination of diagrams without defect. Now consider an arbitrary diagram with defect n. As long as $n \neq 0$, there must exist some dot m with an un-dotted arc above it and no arcs separating them:



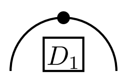
Each box (labeled 1 through 4) is an unspecified diagram with arbitrary defect. By (4.8), this can be rewritten as:



Notice that the defect of m has decreased by 1 in each new diagram. Also notice that the number of un-dotted arcs above diagrams 1 through 4 either decreases by 1 or is unchanged by this relation, so their respective defects either decrease or are unchanged. Thus our diagram with defect n has been rewritten in terms of diagrams with defect $d < n$. By our inductive assumption, each of these can in turn be written as a linear combination of diagrams without defect, so we are done. □

7.2. **Additions that Maintain Normalcy.**

Lemma 7.3. *If D_1 is a normal diagram and D_2 is a completely undotted diagram, then $D_1 \otimes D_2$ and $D_2 \otimes D_1$ are both normal diagrams as well. Additionally, If D_1 is a normal diagram, then the following is also normal:*

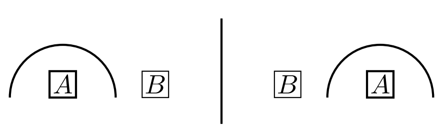


Proof. The tensoring operation, $D_1 \otimes D_2$ or $D_2 \otimes D_1$, does not produce either of the local violations to the definition of normal in Proposition 6.2. The addition of a dotted strand over the entirety of D_1 also does not violate the conditions of normalcy in Proposition 6.2. □

Theorem 7.4. *The space JB^0_{2n} is spanned by the set of all normal JTL diagrams of type $2n \to 0$.*

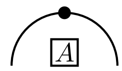
Proof. Let us induct on the value n, where n is the sum of the number of strands and the number of dots. Our base case is when there are no dots on the diagram, which of course results in a normal diagram. Suppose that we can express any diagram with $n < k$ as a linear combination of normal diagrams. Then let us investigate diagrams with $n = k$ strands and dots. By Lemma 7.2, we can write our diagram as a linear combination of diagrams without defects. We can then observe that since the defect of each diagram is 0, if there exists a dot in a diagram, then every arc above it must also have a dot. Thus we can look at the topmost arcs of the diagram, which gives us 3 cases:

Case 1: The leftmost or the rightmost top arc has no dot.



Then we can express the diagram as a tensor product of B and the rest of the diagram. Then we know that as B has value $n \leq k - 1$ and by induction can be expressed as a linear combination of normal diagrams, and by Lemma 7.3 the entire diagram can be expressed as a linear combination of normal diagrams.

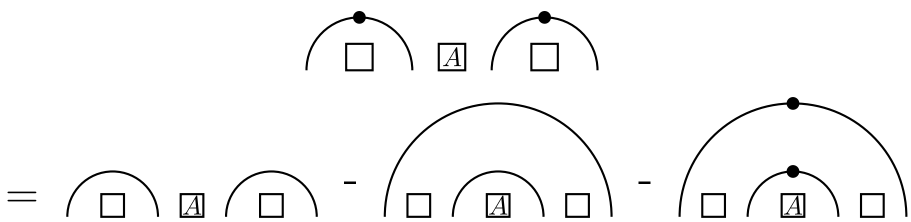
Case 2: The leftmost and rightmost arcs are the same and have a dot.



Then as A has $n = k - 2$, it can be written as a linear combination of normal diagrams by induction, and thus by Lemma 7.3 the entire diagram can be written as a linear combination of normal diagrams.

Case 3: The leftmost and rightmost top arcs are distinct, different, and dotted.

Then we can use one of our identities (4.7)to express it as a linear combination:



In the first two diagrams there are 2 fewer dots, and thus by induction can be expressed as a linear combination of normal diagrams. By case 2, the third diagram can be can be written as a linear combination of normal diagrams, and thus any JTL diagram can be expressed as a linear combination of normal diagrams.

□

Corollary 7.5. *The set of normal JTL diagrams of type* $2n \to 0$ *form a basis for the space* JB^0_{2n}

Proof. It falls from (1.4) and (1.5) that $\dim JB^0_{2n} = \binom{2n}{n}$. By Proposition 6.3 and the previous theorem, normal JTL diagrams provide a spanning set of the right size. Hence they are a basis for JB^0_{2n}. □

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