The Motzkin Category

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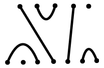
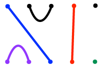
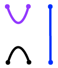
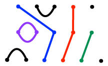
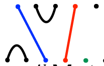
Introduction

Recently, Tom Halverson and Georgia Benkart introduced Motzkin diagrams in [1]. The following is a Motzkin diagram of type $7\to 5$:



which is a morphism in $Hom_{\mathcal{M}}(7,5)$. The Motzkin Category $\mathcal{M}$ consists of diagrams where no more than two vertices are connected and no edges cross. Diagrams can be multiplied vertically as well as horizontally by the operations $\bullet$ and $\otimes$ respectively. Note that $\mathcal{M}$ is closed under both operations. Below illustrates the operations on Motzkin diagrams:

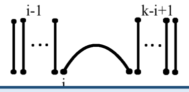
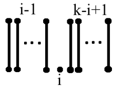
A =  B =  C = 

A $\bullet$ (B $\otimes$ C) =  = 

Katch, Ly, and Posner gave a presentation of Motzkin monoids in terms of generators and relations in [2]. Our main result is a presentation of the Motzkin category using only three generators and five relations.

Normal Form in $\mathcal{M}$

We desire to define a normal form for any Motzkin diagram $D: m+n\to 0$. Consider

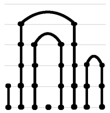
$D_i^{(k)}=$  and $B_i^{(k)}=$ .

Lemma 1

Any Motzkin Diagram $D: m+n\to 0$ can be written uniquely in the form:

$D = (D_{i_1}^{(0)}\bullet D_{i_2}^{(2)}\bullet\cdots\bullet D_{i_k}^{(m+n-l-2)})\bullet(B_{j_1}^{(m+n-l)}\bullet B_{j_2}^{(m+n-l+1)}\bullet\cdots\bullet B_{j_l}^{(m+n-1)})$

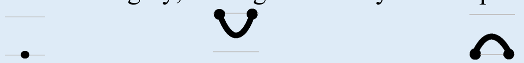
where l is the number of parts with one element, $1=i_1<i_2<\cdots<i_k$, and $j_1<j_2<\cdots<j_l$.

 = 

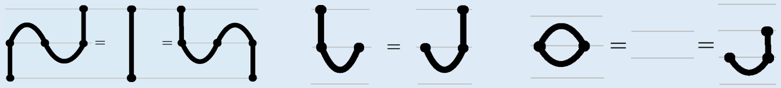
The normal form for the diagram above is $(D_1^{(0)}\bullet D_2^{(2)}\bullet D_5^{(4)})\bullet(B_1^{(6)}\bullet B_4^{(7)})$.

Theorem

As a strict monoidal category, $\mathcal{M}$ is generated by the morphisms



subject only to the relations



The Category $\mathcal{FM}$

In order to prove the theorem, we let $\mathcal{FM}$ denote the strict monoidal category generated by a single object and three morphisms $b: 1\to 0$, $c: 0\to 2$, and $d: 2\to 0$ subject to the relations:

$$(d\odot 1_1)\circ(1_1\odot c) = 1_1 = (1_1\odot d)\circ(c\odot 1_1)$$

$$(1_1\odot b)\circ c = (b\odot 1_1)\circ c \qquad d\circ c = 1_0 = b\circ(b\odot 1_1)\circ c$$

Consider defining $d_i^{(k)} = 1_{i-1}\odot d\odot 1_{k-i+1}$, $b_i^{(k)} = 1_{i-1}\odot b\odot 1_{k-i+1}$, and $c_i^{(k)} = 1_{i-1}\odot c\odot 1_{k-i+1}$. Morphisms in the category $\mathcal{FM}$ are combinations of $b_i^{(k)}$, $c_i^{(k)}$, and $d_i^{(k)}$. Essentially, given a morphism in $Hom_{FM}(m+n,0)$ we need to be able to mold the morphism to a consistent form. We derive the following relations in order to obtain a normal form for our morphisms:

$$d_i^{(k)}\circ c_i^{(k)} = 1_k \qquad d_i^{(k)}\circ c_{i+1}^{(k)} = 1_{k+1} \qquad d_i^{(k)}\circ c_{i-1}^{(k)} = 1_{k+1}$$

$$d_i^{(k)}\circ c_j^{(k)} = c_{j-2}^{(k-2)}\circ d_i^{(k-2)} \qquad d_i^{(k)}\circ c_j^{(k)} = c_j^{(k-2)}\circ d_{i-2}^{(k-2)}$$

$$d_i^{(k)}\circ d_j^{(k+2)} = d_j^{(k)}\circ d_{i+2}^{(k+2)} \qquad c_i^{(k)}\circ c_j^{(k-2)} = c_{j+2}^{(k)}\circ c_i^{(k-2)}$$

$$b_i^{(k-1)}\circ b_j^{(k)} = b_{j-1}^{(k-1)}\circ b_i^{(k)} \qquad b_i^{(k)}\circ d_j^{(k+1)} = d_j^{(k)}\circ b_{i+2}^{(k+2)}$$

$$b_i^{(k)}\circ d_j^{(k+1)} = d_{i-1}^{(k)}\circ b_j^{(k+2)} \qquad c_i^{(k)}\circ b_j^{(k)} = b_j^{(k+2)}\circ c_{i+1}^{(k+1)}$$

$$c_i^{(k)}\circ b_j^{(k)} = b_{j+2}^{(k+2)}\circ c_i^{(k+1)}$$

Lemma 2

Any morphism d in $Hom_{FM}(m+n,0)$ can be written in the form:

$d = \left(d_{i_1}^{(0)}\circ d_{i_2}^{(2)}\circ\cdots\circ d_{i_k}^{(m+n-l-2)}\right)\circ\left(b_{j_1}^{(m+n-l)}\circ b_{j_2}^{(m+n-l+1)}\circ\cdots\circ b_{j_l}^{(m+n-1)}\right)$.

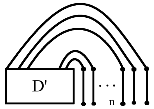
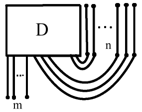
where $1=i_1<i_2<\cdots<i_k$, and $j_1<j_2<\cdots<j_l$.

For example, the morphism $d_1^{(0)}\circ d_2^{(2)}\circ b_1^{(4)}\circ d_3^{(5)}\circ c_4^{(5)}$ can be put into normal form using the relation $d_i^{(k)}\circ c_{i+1}^{(k)} = 1_k$.

Outline of Theorem's Proof

The defining relations of $\mathcal{FM}$ hold among the corresponding diagrams of $\mathcal{M}$, so there is a strict monoidal functor $F:\mathcal{FM}\longrightarrow\mathcal{M}$. To show that $\mathcal{FM}\cong\mathcal{M}$ we must show that F is a bijection.

Let $D: m+n\to 0$ and $D': m\to n$. We have defined a diagrammatic bijection ψ as follows:

$\psi(D)=$  $\psi^{-1}(D')=$ 

Essentially, this bijection is simply a way to get a $m\to n$ diagram from a $m+n\to 0$ diagram and vice versa. Next, we describe an analog to the bijection ψ for the category $\mathcal{FM}$. For morphisms $x: m+n\to 0$ and $y: m\to n$ in $\mathcal{FM}$ define the bijection ϕ as follows:

$\phi(y) = \delta_n\circ(y\odot 1_n)$ and $\phi^{-1}(x) = (x\odot 1_n)\circ(1_m\odot\gamma_n)$ where $\gamma_n = c_n^{(2n-2)}\circ\gamma_{n-1}$ and $\delta_n = \delta_{n-1}\circ d_n^{(2n-2)}$.

Next, consider

$$\begin{array}{ccc} Hom_{FM}(m,n) & \xrightarrow{F} & Hom_{M}(m,n) \\ \downarrow\phi & & \downarrow\psi \\ Hom_{FM}(m+n,0) & \xrightarrow{F} & Hom_{M}(m+n,0) \end{array}$$

The diagram above is commutative and so we only have to show F is a bijection on morphisms of type $m+n\to 0$. Observe $F\left(d_i^{(k)}\right) = D_i^{(k)}$ and $F\left(b_i^{(k)}\right) = B_i^{(k)}$. Since we determined that every Motzkin diagram can be written in the described normal form it follows that F is surjective. Moreover, because the normal form is unique, F is injective. Therefore, $\mathcal{FM}\cong\mathcal{M}$.

References

[1] Benkart, Halverson *Motzkin Algerbras* 2011

[2] Hatch, Ly, Posner *Presentation of the Motzkin Monoid 2013*

Acknowledgements

I owe all my sentiments and intrigue for this topic to Dr. Comes. Thank you for your commitment and patience.