# An Exploration of Knot Groups Using the GAP Programming Language

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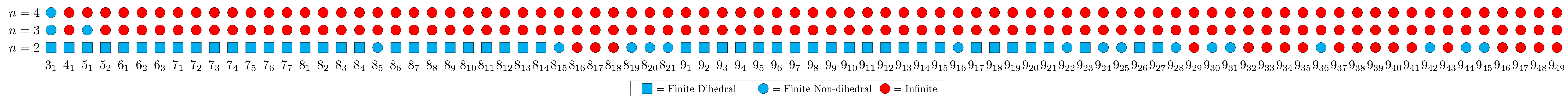



**Fig. 1:** Table depicting the structure of the quotients of the form **G**/<$a^n$>, where **G** is a knot group. Knots are labeled using Alexander-Briggs notation. Groups were considered Infinite if GAP ran out of memory before calculating the size of the group.

## Introduction

Knot Theory is a subject concerned with the analysis of knots. To a knot theorist, a knot results from taking an infinitely thin length of string, tangling it up, and then fusing the ends of the string together to close the string into a single loop. Knot Theory has been applied to other sciences, particularly Biochemistry and Synthetic Chemistry. These applications include:

- Modeling the action of DNA Topoisomerases
- Synthesis of knotted molecules
- Analyzing topological stereoisomers.

There are properties of knots, called knot invariants, which are useful for telling if two knots are equivalent (i.e. one is just a tangled up version of another). If two knots differ at any single knot invariant, then they are not equivalent. One such invariant is a knot group. By utilizing GAP, a programming language built for computational discrete algebra, we were able to write scripts that automatically generated and analyzed quotients of knot groups.

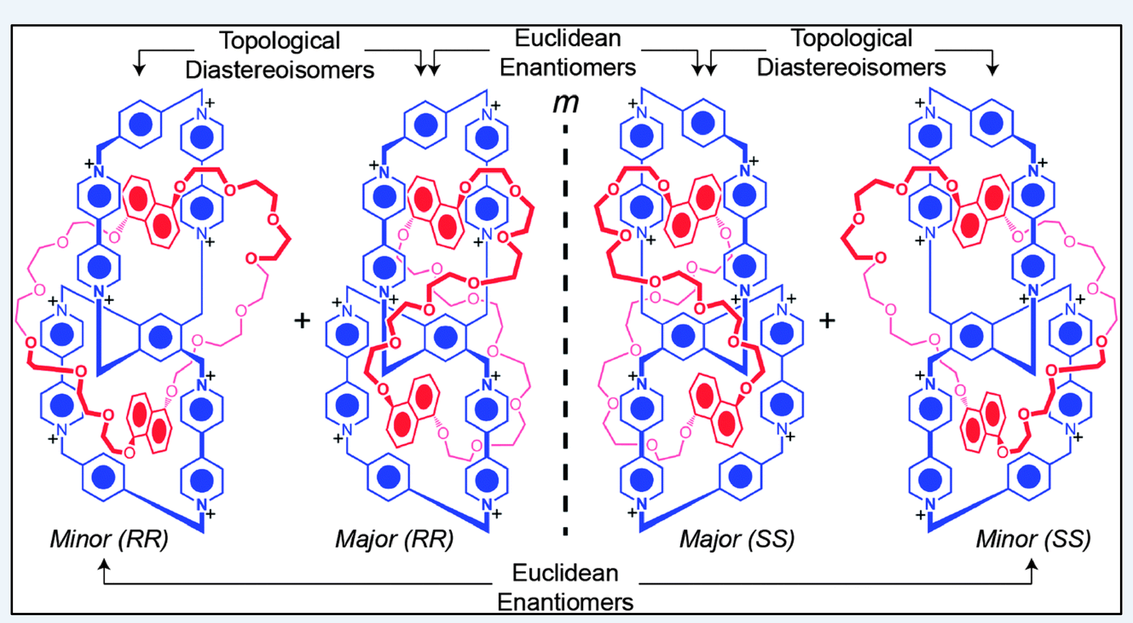
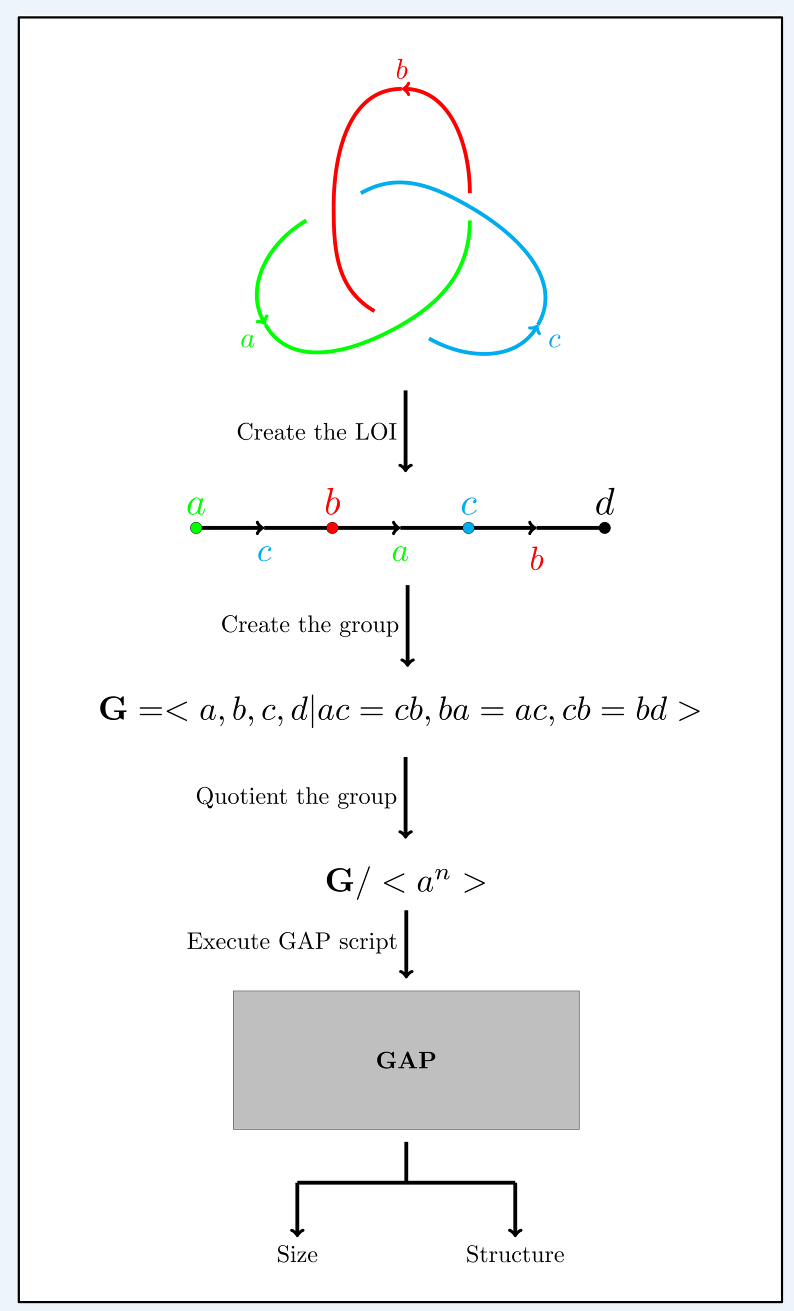



**Fig. 2:** An example of a molecule with topological stereoisomers. Figure from *Topological Isomerism in Chiral Handcuff Catenane.*[1]

## The GAP Programming Language

GAP (Groups, Algorithms, Programming) is a system that was developed to perform computational discrete algebra. It is essentially a programming language that comes with a library of many algebraic structures, as well as a library of functions that can mathematically analyze such structures. GAP allowed for the writing of powerful scripts that could automatically characterize the quotients of knot groups we generated, giving us the size and structure of the group.

## Methods




## Topological Knot Groups

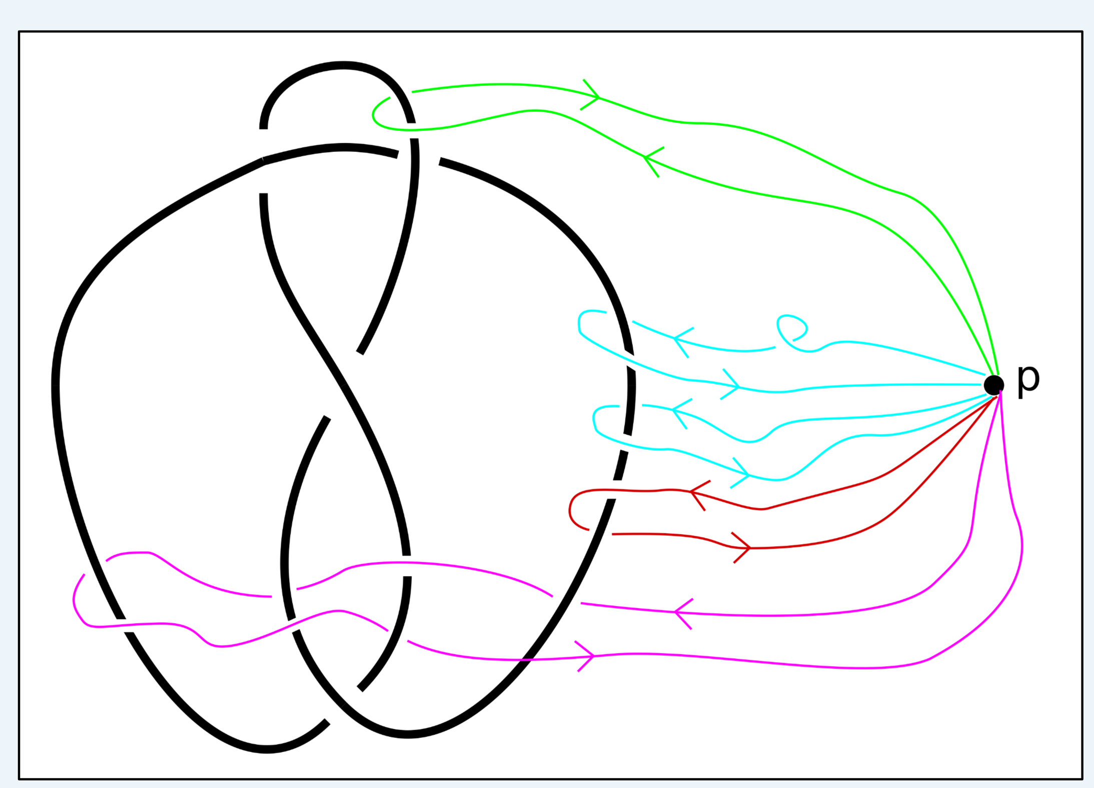



**Fig. 3:** These colored paths are elements of the knot group for the figure-eight knot (AKA $4_1$). The two blue paths are equivalent to one another, as they have the same orientation and can be smoothly deformed between each other without passing through the knot.

## Wirtinger Presentations

Wirtinger presentations of knot groups are easier to work with computationally and are known to be equivalent to the topological presentation. Wirtinger presentations construct knot groups using the strands and crossings to create a set of generators and relations. Note that, in the example below the paths ab and bc are equivalent. This is homologous to the relation ab = bc, which would exist in the Wirtinger presentation of a knot with such a crossing.

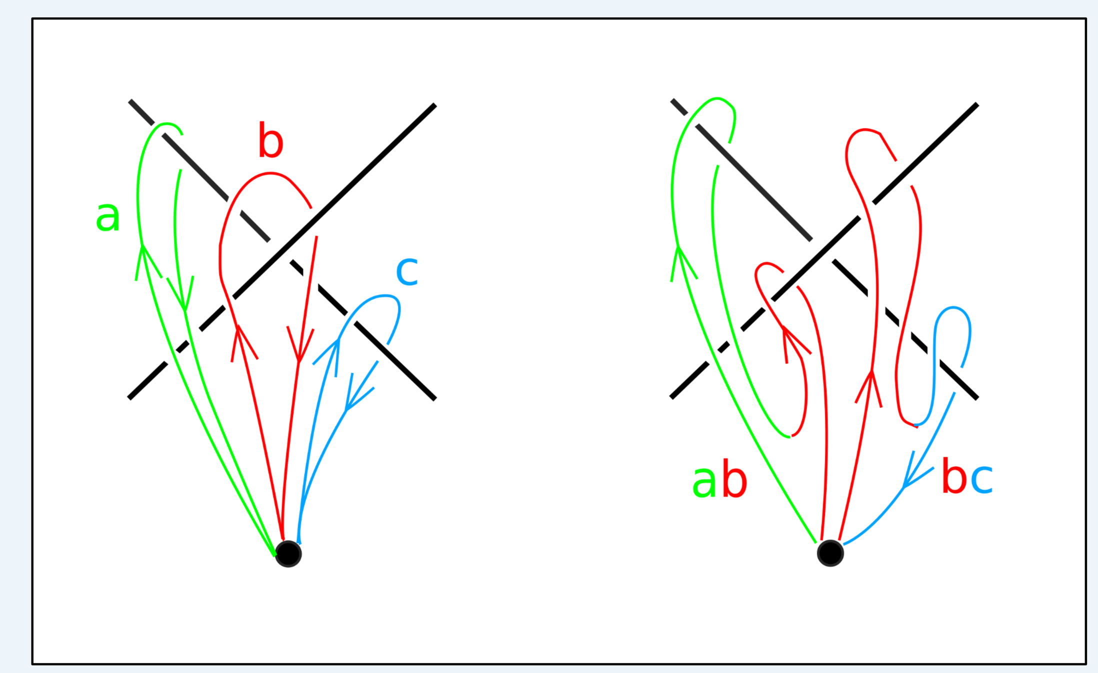



**Fig. 4:** Example of how a Wirtinger presentation relates to the topological presentation of the knot group.

As an example, consider the group

$$\mathbf{G} =< a, b, c, d | ac = cb, ba = ac, cd = da >$$

which is the Wirtinger presentation of the knot group for the trefoil knot ($3_1$). The generators (left of the bar) are the symbols we're allowed to use, and the relations (right of the bar) give us algebraic rules to simplify strings of symbols in this group. If we add the additional relation $a^2 = 1$, then we get the quotient group for $3_1$ where n = 2. For example the string "aacba" in the quotient group can be simplified as follows:

$$\begin{aligned} aacba &= a^2cba \\ &= cba \qquad (a^2 = 1) \\ &= aca \qquad (ac = cb) \\ &= baa \qquad (ba = ac) \\ &= b \qquad (a^2 = 1) \end{aligned}$$

## Discussion

- It is well known that the knot group of the trefoil knot is isomorphic to the so-called Braid group $B_3$. Our results agree with Coxeter's results on the quotients of braid groups.[2]
- The majority of quotients for n = 2 are finite and dihedral
- The complexity and size of the group increases with minimal crossing number
- Nearly all quotients for n = 3 and n = 4 are infinite, with the exception of small torus knots

## Future Work

- Determine the conditions of the knot or LOI that cause the quotient group to be dihedral when n = 2.
- Examine quotients of virtual knots: LOI's that don't correspond to a physical knot.
- Examine particular families of knots in more detail. Of particular interest are the torus knots and twist knots, as their LOI's can be generated via GAP rather than being done by hand.

## Acknowledgements

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This work was inspired by Dr. Harlander's REU.[3]

## References

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