The Balanced Oriented Partition Diagram Category and the General Permutation Group

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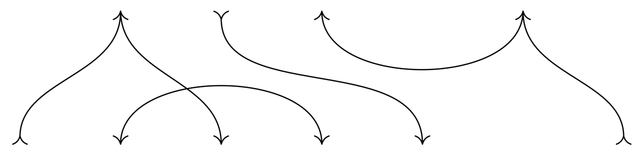
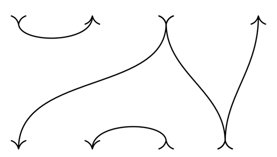
Balanced Oriented Partition Diagrams BOP

An oriented partition diagram consists of two rows of vertices that are either up arrows or down arrows and are of type $a \to b$ where a and b are strings of arrows. For example,



is of type $\uparrow\downarrow\downarrow\downarrow\downarrow\uparrow \to \uparrow\downarrow\uparrow\uparrow$. A *part* of one of these diagrams consists of all vertices connected to one another, so this diagram has 4 parts. A vertex is considered an *in* if the arrow points in toward the center of the diagram, and an *out* if the the arrow point away from the center of the diagram, i.e. an up arrow on the top row is an out while an up arrow on the bottom row is an in.

An oriented partition diagram is considered *balanced* when each part contains an equal number of ins and outs. For example the previous diagram is not balanced since 2 parts do not have an equal number of ins and outs. However, the diagram

$D =$  (1)

is balanced, since each part contains an equal number of ins and outs. Balanced oriented partition diagrams are the morphisms of the balanced oriented partition diagram category, BOP. The previous diagram is in $\mathrm{Hom}_{BOP}(\downarrow\downarrow\uparrow\uparrow, \downarrow\uparrow\downarrow\uparrow)$.

Generalized Permutation Matrices Γ_d

Γ_d is the group of generalized permutation matrices and its elements are $d \times d$ square matrices where each row and column contain exactly one nonzero entry. An element $\gamma \in \Gamma_d$ can be expressed as $\sigma \cdot diag(\gamma_1, \gamma_2, \ldots, \gamma_d)$ where σ is a permutation matrix, which is a square matrix where every entry is either a 1 or a 0 and each row and column contain one entry of 1, and $\gamma_1, \gamma_2, \ldots, \gamma_d$ are complex numbers. For example,

$$\begin{bmatrix} 0 & \pi & 0 \\ 0 & 0 & 11 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 11 \end{bmatrix} = \sigma \cdot diag(2, \pi, 11)$$

Γ_d acts on the vector space $V = \mathbb{C}^d$ and its dual V^* in the following ways: Let γ be as above, let v_i be some basis vector of V, and let f_i be the dual basis vector of V^*. Then Γ_d acts on V by $\gamma \cdot v_i = \gamma_i v_{\sigma(i)}$ and acts on V^* by $\gamma \cdot f_i = \frac{1}{\gamma_i} f_{\sigma(i)}$. On tensor products, Γ_d acts in the following way:

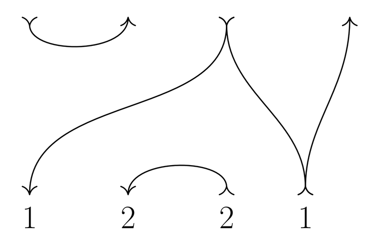
$$\begin{aligned} \gamma \cdot (v_i \otimes f_j \otimes f_k \otimes v_j) &= \gamma_i v_{\sigma(i)} \otimes \frac{1}{\gamma_j} f_{\sigma(j)} \otimes \frac{1}{\gamma_k} f_{\sigma(k)} \otimes \gamma_j v_{\sigma(j)} \\ &= \frac{\gamma_i}{\gamma_k} (v_{\sigma(i)} \otimes f_{\sigma(j)} \otimes f_{\sigma(k)} \otimes v_{\sigma(j)}). \end{aligned}$$

This group action gives V, V^*, and their tensors the structure of a representation of Γ_d. A Γ_d-linear map is a function $\varphi : U \to W$ between two representations of Γ_d such that φ is linear over $\mathbb{C}$ and $\varphi(\gamma \cdot u) = g \cdot \varphi(u)$ for all $\gamma \in \Gamma_d$ and $u \in U$. Let $\mathrm{Rep}(\Gamma_d)$ denote the category of representations of Γ_d and Γ_d-linear maps between them.

The Functor $F : BOP \to \mathrm{Rep}(\Gamma_d)$

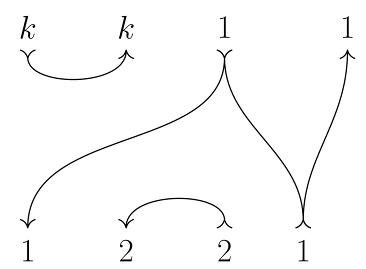
Let $F : BOP \to \mathrm{Rep}(\Gamma_d)$ be a functor where Ob $BOP \to$ Ob $\mathrm{Rep}(\Gamma_d)$ is given by $a \mapsto V^a$ where a is some string of arrows (denoted $a \in \langle\uparrow, \downarrow\rangle$). We map diagrams in the balanced oriented partition category to maps in $\mathrm{Rep}(\Gamma_d)$ by following process:

Let D be the balanced partition diagram given before. It is of type: $\downarrow\downarrow\uparrow\uparrow \to \downarrow\uparrow\downarrow\uparrow$, so the function that we find in $\mathrm{Rep}(\Gamma_d)$ will be of the type $V^{\downarrow\uparrow\downarrow\uparrow} \to V^{\downarrow\uparrow\downarrow\uparrow}$. Each up arrow represents the vector space $V = \mathbb{C}^d$ and each down arrow represents its dual V^*, so we will find a function $g_D : V^* \otimes V^* \otimes V \otimes V \to V^* \otimes V \otimes V^* \otimes V$. Now we will "color" the diagram by labeling bottom vertices with different numbers $1, \ldots, d$ such that all connected vertices have the same label.



(2)

Now, for any top vertices that are connected to a bottom vertex, we color it accordingly. If a top vertex is not connected to a bottom vertex, its color is not determined by the bottom vertices, so we label it with a variable, once again labeling connected vertices the same. In this case we will label our undetermined top vertices k. Assuming $d = 2$, k could either be 1 or 2.



(3)

Then g_D maps tensor product of basis vectors

$$f_1 \otimes f_2 \otimes v_2 \otimes v_1 \mapsto f_1 \otimes v_1 \otimes f_1 \otimes v_1 + f_2 \otimes v_2 \otimes f_1 \otimes v_1.$$

Since we know where basis elements go, we can extend linearly to any element of $V^* \otimes V^* \otimes V \otimes V$.

Main Result

The main result of this research was showing that any Γ_d-linear map between tensor products of V and V^* can be obtained through balanced oriented partition diagrams.

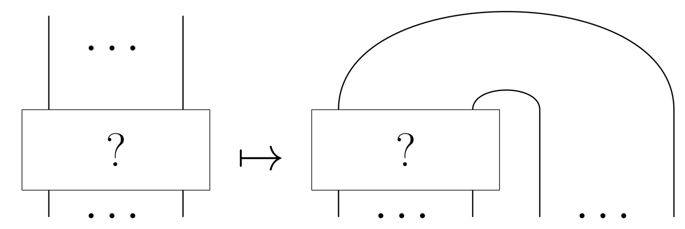
To state this formally:

Theorem. The functor $F : BOP \to \mathrm{Rep}(\Gamma_d)$ is full.

Outline of Proof

We need to show that the map $\mathrm{Hom}_{BOP}(a, b) \to \mathrm{Hom}_{\Gamma_d}(V^a, V^b)$, where $a, b \in \langle\uparrow, \downarrow\rangle$, is surjective.

There is a bijection $\mathrm{Hom}_{BOP}(a, b) \to \mathrm{Hom}_{BOP}(ab^*, 0)$ where for $b = b_1 b_2 \ldots b_n$ $b^* = b_n^* \ldots b_1^*$ with $\uparrow^* = \downarrow$, given by the diagram below.



(4)

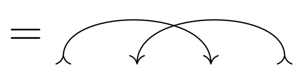
Then we know that $\mathrm{Hom}_{BOP}(ab^*, 0) \to \mathrm{Hom}_{\Gamma_d}(V^{ab*}, \mathbb{C})$ is surjective:
Let an orbit O be a balanced orbit and define

$$f_O(\vec{v}) = \begin{cases} 1 & \vec{v} \in O \\ 0 & \text{otherwise} \end{cases}$$

For example, if $O_1 = \{v_i \otimes f_j \otimes f_i \otimes v_j : i \neq j\}$, then

$$f_{O_1}(\vec{v}) = \begin{cases} 1 & \text{if } \vec{v} = v_i \otimes f_j \otimes f_i \otimes v_j, i \neq j \\ 0 & \text{otherwise} \end{cases}$$

Then f_{O_1} corresponds to the diagram

$D_1 =$  (5)

Though the two correspond, $F(D_1) \neq f_{O_1}$, but f_{O_1} is achieved through a combination of D_1 and other diagrams related to D_1.

In our proof we generalize to any orbit, so for any f_O we can find a combination of diagrams X_D such that $F(X_D) = f_O$. We can show that the $\{f_O\}$ is a basis for $\mathrm{Hom}_{\Gamma_d}(V^{ab^*}, \mathbb{C})$, so for any function $g \in \mathrm{Hom}_{\Gamma_d}(V^{ab^*}, \mathbb{C})$, we find it as a unique linear combination of f_O which in turn can be found in terms of $F(X_D)$ for $D \in \mathrm{Hom}_{BOP}(ab^*, 0)$.

There exists a bijection $\mathrm{Hom}_{\Gamma_d}(V^{ab^*}, \mathbb{C}) \to \mathrm{Hom}_{\Gamma_d}(V^a, V^b)$ by applying F to the bijection described above. Hence, $\mathrm{Hom}_{BOP}(a, b) \to \mathrm{Hom}_{\Gamma_d}(V^a, V^b)$ is surjective.

References

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