$\sqrt[n]{2}$-Binary Look and Say Sequences

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Conway's Look and Say

The look and say sequence was first introduced by John Conway. Roughly speaking, the rule for generating the sequence is "say what you see".

Let's look at a standard look and say sequence:

$$11 \to 21 \to 1211 \to 111221 \to 312211 \to 13112221 \to \cdots$$

Conway estimated the ratio of the lengths of terms also called Conway's constant to approach

$$1.303577269\ldots$$

Conway discovered a beautiful structure of these look and say sequences that is ultimately governed by a linear transformation of a 92-dimensional vector space!

$\sqrt[n]{2}$-Binary Number System

We will write $[m]_n$ for $\sqrt[n]{2}$-binary representation of any integer m. When $n = 2$

$$12 = 1(\sqrt{2})^6 + 0(\sqrt{2})^5 + 1(\sqrt{2})^4 + 0(\sqrt{2})^3 + 0(\sqrt{2})^2 + 0(\sqrt{2})^1 + 0(\sqrt{2})^0$$

$$[12]_2 = 1010000$$

let's look at some more conversions:

Decimal	Binary	$\sqrt{2}$-Binary	$\sqrt[3]{2}$-Binary	$\sqrt[4]{2}$-Binary
0	0	0	0	0
1	1	1	1	1
2	10	100	1000	10000
3	11	101	1001	10001
4	100	10000	1000000	100000000
5	101	10001	1000001	100000001

From the conversions above we can see that $\sqrt[n]{2}$-binary representations are obtained by inserting $n-1$ 0's between each bit in binary representation. We can write $\sqrt[n]{2}$-binary representations in exponent form. The exponents determine the number of consecutive 0's or 1's. For example $1^3 0^2$ would represent 3 consecutive 1s followed by 2 consecutive 0's i.e 11100.

The Split Method

The splitting method is crucial to explaining look and say sequences. Let's look at a $\sqrt{2}$-binary look and say sequence with seed 0:

$$0 \to 10 \to 1110 \to 101110 \to 1110101110 \to \cdots$$

The sequence splits right between 0 and 1. The fourth term in the sequence is a combination of elements 10 and 1110. Meanwhile, the fifth term in the sequence is made up of elements 10 and two 1110's.

$\sqrt[n]{2}$-Binary Look and Say Sequence With Seed 0

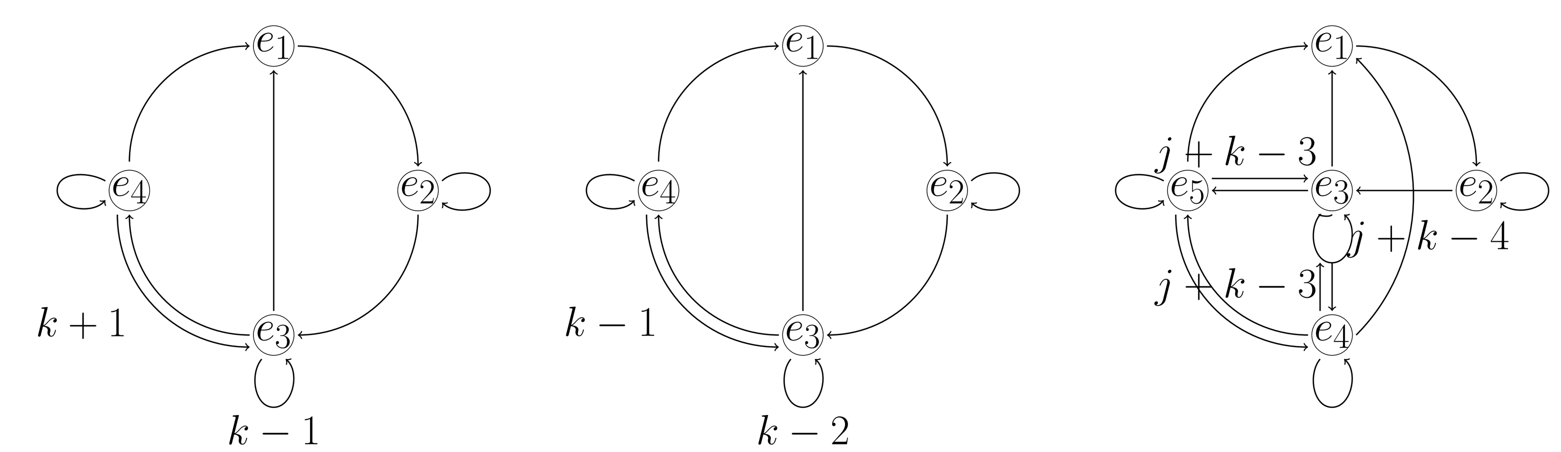
Suppose $n > 2$ has the binary form $1^j 0^k$. Then the $\sqrt[n]{2}$-binary look and say sequence with seed 0 splits into the given elements

Case 1: When $j = 1$, the frequent elements are $e_1 = 10, e_2 = 1^3 0, e_3 = 10^{n-1}, e_4 = 1^3 0^{n-1}$, which decay as $e_1 \to e_2$, $e_2 \to e_3 e_2$, $e_3 \to e_4 e_3^{k-2} e_1$, $e_4 \to e_3 e_4 e_3^{k-2} e_1$.

Case 2: When $j = 2$, the frequent elements are $e_1 = 10$, $e_2 = 1^3 0$, $e_3 = 10^{n-1}$, and $e_4 = 1^3 0^{2n-1}$, which decay as $e_1 \to e_2, e_2 \to e_3 e_2, e_3 \to e_4 e_3^{k-1} e_1$, and $e_4 \to e_3^{k+1} e_4 e_1$.

Case 3: When $j > 2$, the frequent elements are $e_1 = 10$, $e_2 = 1^3 0$, $e_3 = 10^{n-1}$, $e_4 = 10^{2n-1}$ and $e_5 = 1^3 0^{n-1}$, which decay as $e_1 \to e_2, e_2 \to e_3 e_2, e_3 \to e_5 e_3^{j+k-4} e_4 e_1$, $e_4 \to e_5 e_3^{j+k-3} e_4 e_1$ and $e_5 \to e_3^{j+k-3} e_5 e_4 e_1$.

Decay Graph



Case 1   Case 2   Case 3

Characteristic Polynomials and Growth Rates

To estimate the growth rate of the sequence, we can encode the frequent elements into a $k \times k$ decay matrix where k is the total number of frequent elements. The characteristic polynomials that we obtain from decay matrices gives us eigenvalues. The ratio of the length of the terms of the look and say sequence approaches the maximal real eigenvalue.

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & k-2 & k-1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & k-1 & k+1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & j+k-4 & j+k-3 & j+k-3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Case 1   Case 2   Case 3

Characteristic Polynomials:
Case 1: $-\lambda^2(-\lambda^2 + k\lambda + 2 - k)$
Case 2: $\lambda^4 - \lambda^3 - 2\lambda^2 - \lambda^3 + k\lambda^2 + \lambda$.
Case 3: $j\lambda^4 - j\lambda^3 - \lambda^5 - \lambda^4 + 4\lambda^3 - \lambda^2 + k\lambda^4 - k\lambda^3$

We find that the maximal real eigenvalue for case 1 is $\lambda = \dfrac{k + \sqrt{k^2 - 4k + 8}}{2}$.

In case 2, $\lambda = \dfrac{\alpha}{3\sqrt[3]{2}} - \dfrac{\sqrt[3]{2}(-k^2 + k - 7)}{3\alpha} + \dfrac{k+1}{3}$ where

$$\alpha = \sqrt[3]{(2k^3 - 3k^2 + 3\sqrt{3}\sqrt{-k^4 + 2k^3 - 15k^2 + 14k - 49} + 15k - 7}$$

Cosmological Lemma

We refer to the terms as n-days-old if they appear after n^{th} decay in the sequence. We prove that no more than 3 consecutive 1's will appear in the $\sqrt[n]{2}$-binary look and say sequence starting from day one. Additionally, the number of consecutive 0's in a two day-old string are limited to $1, n$, and $in \pm 1$ for $i \in \mathbb{N}$.

Cosmological Lemma: The decay of any one day old element $1^y 0^z$ will have less than z consecutive 0's whenever $z^n < 2^{z-1}$ and $z > n$.

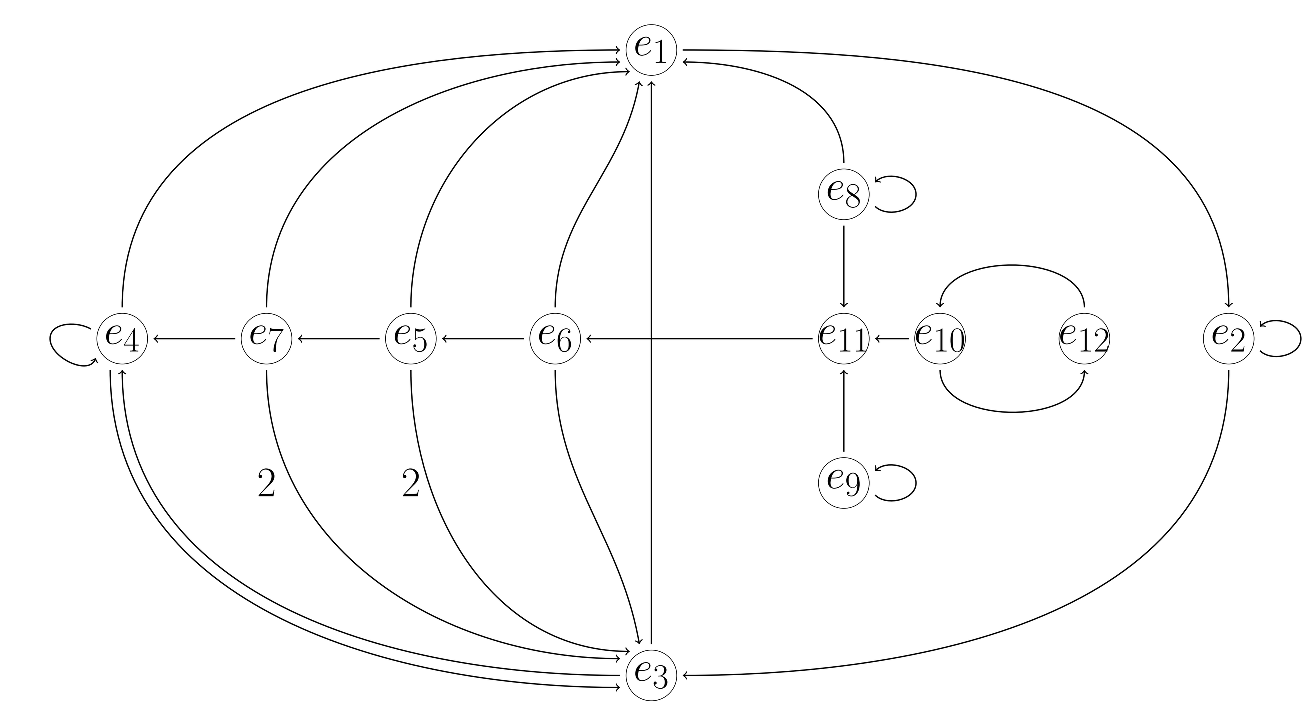
Cosmological Theorem

The Cosmological Lemma allows us to determine finite number of seeds that one needs to check to find all persistent elements in any $\sqrt[n]{2}$-binary look and say sequence. The following theorem is an example.

Cosmological Theorem for n=4: Terms of any $\sqrt[4]{2}$-binary look and say sequence are eventually compounds of the 12 elements found in the table. Relative abundances and decay of the sequence is also found below.

Element	String	Abundances	Decay
e_1	10	25	e_2
e_2	$1^3 0$	25	$e_3 e_2$
e_3	10^3	25	$e_4 e_1$
e_4	$1^3 0^3$	25	$e_3 e_4 e_1$

Element	String	Abundances	Decay
e_5	$1^3 0^{11}$	0	$e_3 e_7 e_3 e_1$
e_6	$1^3 0^9$	0	$e_3 e_5 e_1$
e_7	$1^3 0^7$	0	$e_3 e_4 e_3 e_1$
e_8	$1^2 0^3$	0	$e_{11} e_8 e_1$
e_9	$1^2 0$	0	$e_{11} e_9$
e_{10}	1^2	0	$e_{11} e_{12}$
e_{11}	10^4	0	e_6
e_{12}	1	0	e_{10}



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